A TEXTBOOK ON THE TEACHING OF ARITHMETIC

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STAMPER TEACHING ARITH.

W. P. I

PREFACE

In planning the present book, the author has kept in mind three aspects of the teacher's preparation: the need of the perspective gained through a comprehension of the larger teaching problems concerned in the teaching of arithmetic, the need of a clear insight into the content of arithmetic, and the need of an understanding of arithmetic method. The book exploits no new doctrine. It lays no claim to completeness. It has been prepared with the primary aim of supplying the practical needs of prospective teachers, or teachers new in the service. On account of the effort made to refresh the mind of the reader in the subject matter of arithmetic, the reading frequently goes beyond the requirements of the grades, this being especially noticeable in the treatment of percentage and in the solution of problems. If any dogmatism appears in the context, it is the result of efforts to be clear and convincing.

The author wishes to take this opportunity to thank those who kindly read the manuscript of this book, and offered helpful criticisms.

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THE TEACHING OF ARITHMETIC

CHAPTER I

WITH REFERENCE TO THE HISTORY OF ARITHMETIC

Counting. The analogies between the development of the child and that of the race have often been pointed out. Just as the child first learns to count when he begins his study of arithmetic, so primitive man began his arithmetical development. A primitive race does not at first need many numbers in counting, the expression "a great many" sufficing for the higher numbers. In our numeration to-day we count by tens, but this is not absolutely necessary, although apparently it is a great convenience. Most of the early nations counted by tens, since it was easy to count off ten on the fingers of the two hands.

Other scales than ten. History tells us of tribes and races counting by fives, sevens, elevens, and twenties. It is a common practice to-day to count by fives by making four marks and drawing a fifth across them and continuing this process throughout the counting. We have probably followed tradition in counting by twos when we speak of spans of horses, braces of birds, and the like. We are essentially counting by twelves or by twenties in enumerating so many dozen or score. In using a scale of ten, we

first count through ten (the base) and then say, after twelve, the equivalent of:

ten and three, ten and four, . . . , ten and nine, two tens;

two tens and one, two tens and two, . . . , three tens; etc.

A nation that counted relatively large numbers in the scale of five would have said the equivalent of:

one, two, three, four, five;

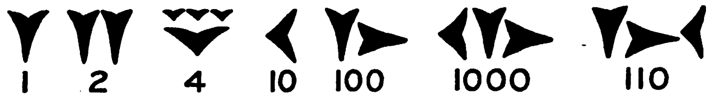
five and one, five and two, . . . , five and four, two fives;

two fives and one, . . . , two fives and four, three fives, etc.

Number symbols. Number symbols were used by primitive people almost as soon, perhaps, as they formulated a system of counting. The most primitive method was to represent numbers by strokes. One stroke | for *one*, two strokes | | for *two*, and so on. The earliest Egyptian and Phœnician writings show the numerals from one to ten represented by the corresponding number of strokes. A single symbol was used for ten, and this was repeated the requisite number of times for the multiples of ten.

The additive principle. This additive principle, by which the numbers represented by two adjacent symbols are to be added, was used in part by all nations. In the Egyptian hieroglyphic symbols, 32 was written ∩∩∩||. The Greeks, using the letters of their alphabet, represented 21 by *κα*, where *κ* represented twenty and *α*, one.

The multiplicative principle. The multiplicative principle secures a less complex symbolism, especially in the larger numbers. The use of these two principles is shown in the wedge-shaped symbols used by the Babylonians.



The symbol for 1000 is made by placing the symbol for 10 at the left of the symbol for 100, and, as 1000 is the equivalent of 10×100, we have the multiplicative principle illustrated. The additive principle is illustrated, for example, in the writing of 110, in which the symbol for 10 is placed at the right of the symbol for 100.

The subtractive principle. The Romans used the subtractive principle as well as the additive and the multiplicative. The subtractive principle is illustrated where I is placed at the left of X to obtain 9; the additive principle, where I is placed at the right of X to obtain 11.

I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	2	3	4	5	6	7	8	9	10	11

XL	L	LX	C	D	M	$\overline{\text{X}}$
40	50	60	100	500	1000	10,000

In general, symbols placed respectively at the left or right of symbols representing greater numerical value indicate a subtraction or an addition. The horizontal bar over a group of symbols shows that the number represented by that group is to be multiplied by 1000. For example, XX $=20$ and $\overline{\text{XX}} = 20{,}000$.

We employ the additive principle in our use of the Hindu (Arabic) number symbols, combined with a multiplicative principle. For instance, 21 represents 2 tens plus 1, a multiplication of an understood unit, ten, by 2,

and then an addition. In 321, there are understood two multiplications and the addition of three terms.

The principle of place value in the Hindu system. The greatest contribution to the development of number symbolism was made by the Hindus, who first employed the principle of place value in a scientific way. The invention of a symbol for zero, which is generally accredited to them, was hardly of less importance in that by its use the principle of place value was made effective. In the number 324, the three symbols, or numerals, have *intrinsic* values of *three*, *two*, and *four* respectively. They have also in turn *place*, or *local*, values of *three hundreds*, *two tens*, and *four units*. In 304, the middle symbol indicates that there are no tens. Without a symbol for zero there would be danger of writing this 34.

The spread of the Hindu numerals. The various nations were relatively slow in recognizing the value of the Hindu numerals. The Arabs, who were at one time credited with having invented them, introduced their use in the Asiatic and African seaport towns on the Mediterranean coast, and thereby made them accessible to southern Europe. Leonardo of Pisa, who described the numerals and the Hindu-Arabic methods of calculation in his *Liber Abaci* (1202), is generally recognized as being the first to introduce the Hindu system into Italy. Europe did not, however, immediately adopt the new numerals, for the Roman numerals were still used in computation, although to a much less extent, as late as the sixteenth century.

Fractions. The idea of a fraction was probably not any more difficult to the ancients than simple counting, for a

fraction may be considered as involving two countings: first, a counting of the number of equal units in a measured quantity; second, a counting of a number of these equal units. For instance, in considering three fourths, we first think of a quantity divided into four equal parts, and then count off three of these parts. Complex fractions require, of course, a broader definition.

Symbols for fractions. The invention of a scientific symbolism for fractions naturally followed that for integers. In the Egyptian hieroglyphics $\frac{1}{2}$ was written  and $\frac{2}{3}$  or . In the Egyptian hieratic symbolism all fractions were written with the numerator 1. A fraction like $\frac{2}{7}$ was expressed as the sum of the two unit fractions $\frac{1}{4}$ and $\frac{1}{28}$. Each unit fraction was represented by the symbol for the denominator with a dot above it. The earliest record of the use of fractions by the Egyptians is found in a papyrus now in the British Museum, written by a scribe, Ahmes, about 1700 B.C. This manuscript gives the earliest known account of Egyptian mathematics.

The Babylonians chose a constant denominator 60 for their fractions, while the Romans usually employed 12. The Greeks wrote their fractions from left to right, the numerator followed by the denominator written twice, thus $\iota\zeta'$ $\kappa\alpha''$ $\kappa\alpha''$ for $\frac{17}{21}$. Unit fractions were represented by the denominator written once, the numerator 1 being omitted: δ'' for $\frac{1}{4}$. The Hindus wrote fractions as we write common fractions to-day, but without the line: $\begin{smallmatrix}3\\8\end{smallmatrix}$ for $\frac{3}{8}$.

Notation for decimal fractions. The present notation for decimal fractions, first systematized by Stevin in the latter part of the sixteenth century, was a late invention in the

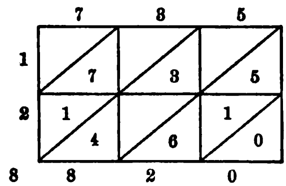
history of arithmetic. This seems especially remarkable when one considers that nations from remote antiquity had used the decimal system with reference to whole numbers, and that the Hindu system had been practically in its present form for about 1000 years. Stevin wrote 237.578 thus — $237_{⓪}5_{①}7_{②}8_{③}$. Pitiscus, in 1612, was the first to use the decimal point.

Calculations without symbols. Nations have performed their arithmetical calculations either with or without the use of written symbols. Primitive nations naturally used the latter method. The earliest calculations were made by counting, either orally or by making marks on sticks or in the sand. These methods sufficed for a primitive people having little need for computation beyond addition and subtraction. It was a common practice to perform calculations by means of pebbles in grooves in the sand or by means of beads on rods, as was done by the Greeks, Romans, and Chinese. The Chinese, Japanese, and Russians of to-day employ this method in their use of the abacus, or swan pan. Some of the nations of Europe, notably the Germans, calculated by means of pennies placed on lines, a device similar in principle to the swan pan. The somewhat complicated system of finger-reckoning was quite common among the orientals.

With symbols. 1. *Subtraction.* Little progress was made by the Babylonians, Egyptians, Greeks, and Romans in the art of calculating, on account of their cumbersome systems of writing numbers. The Hindu system proved usable in calculating, and has become a world system. Two of our methods of subtraction can be traced back to the

Hindus, the so-called borrowing method and an add-to-the-subtrahend method. In subtracting 348 from 821, they said either 8 from 11, 3; 4 from 11, 7; 3 from 7, 4; or 8 from 11, 3; 5 from 12, 7; 4 from 8, 4.

2. *Multiplication.* Multiplication was performed in various forms, even where the Hindu numerals were used. One form used by the Hindus is illustrated in the following example, in which it is required to multiply 735 by 12:



The Hindu, Brahmagupta (seventh century A.D.), multiplied 235 by 288 as follows:

235	2	470
235	8	1880
235	8	1880
		67680

Bhascara, another Hindu, who lived in the twelfth century, gave, in his *Lilavati*, six different ways of multiplying, two of which are illustrated below. The first is closely related to our common form. The second is a usable short method. Let it be required to multiply 135 by 12:

(*a*)	135	135	(*b*)	135	10	1350
	1	2		135	2	270
		270				1620
		135				
		1620				

3. *Division.* A method of dividing that was common in Europe as late as the seventeenth century was the galley

method, so called on account of the resemblance the work bears to a galley, or ship. This resemblance is particularly noticeable where large numbers are involved. The following solution of 857 divided by 39 illustrates the method:

Write the divisor below the dividend and the quotient at the right,

$$(1)\ \begin{array}{ccc|l} 2 & & & \\ \cancel{8} & 5 & 7 & 2 \\ \cancel{3} & 9 & & \end{array} \qquad (2)\ \begin{array}{ccc|l} \cancel{2} & 7 & & \\ \cancel{8} & \cancel{5} & 7 & 2 \\ \cancel{3} & \cancel{9} & 9 & \\ & 3 & & \end{array} \qquad (3)\ \begin{array}{ccc|l} & 3 & & \\ & 4 & & \\ \cancel{2} & \cancel{7} & 8 & \\ \cancel{8} & \cancel{5} & \cancel{7} & 21 \\ \cancel{3} & \cancel{9} & \cancel{9} & \\ & \cancel{3} & & \end{array}$$

The first figure in the quotient is 2. 2 times 3 is 6. 8 less 6 is 2. Cross out the 3 and the 8. Write the remainder 2 above the 8. 2 times 9 is 18. 25 less 18 is 7. Cross out the 9, the 2, and the 5. Write the remainder 7 above the 5. Rewrite the divisor one place to the right, in the position shown in step (2) above. We must now divide 77 by 39. 3 is contained in 7 twice, but 2 will be found too large a quotient, 1 being the correct figure. Once 3 is 3. 7 less 3 is 4. Cross out the 3 and the 7. Once 9 is 9. 47 less 9 is 38. Cross out the 4, 7, and 9. Write the 3 of the 38 above the 4 and the 8 above the 7. The quotient is 21 and the remainder is 38. The method is readily understood when worked in the form we use to-day:

$$\begin{array}{rl} 21 & \\ 39\overline{)857} & \\ \underline{6} & = 2 \times 3 \\ 25 & \\ \underline{18} & = 2 \times 9 \\ 7 & \\ \underline{3} & = 1 \times 3 \\ 47 & \\ \underline{9} & = 1 \times 9 \\ 38 & \end{array}$$

The methods of multiplying and dividing in common use to-day were employed in Europe as early as the fifteenth

century. They were the result of an evolution from the Hindu-Arabic methods.

Other mathematical subjects. The beginnings of algebra are found in the Ahmes papyrus, where use is made of an unknown quantity in the solution of problems, but the science of algebra originated in the fourth century B.C. in the Greek city of Alexandria in Egypt. The Hindus, later, contributed especially to the solution of equations and the theory of irrationals. Modern algebra dates from the sixteenth century.

Practical geometry dates back to ancient Egypt. The Ahmes papyrus gives examples in mensuration. Logical geometry was systematized by the Greeks during a period of 300 years, beginning about 600 B.C.

Trigonometry had its initial development under Greek influence at Alexandria from the second century B.C. to the second century A.D.

The higher branches of mathematics have been developed since the seventeenth century.

Instruction in arithmetic. The study of arithmetic was considered of little importance in the early history of Europe. In the schools of the Middle Ages it was taught for its value in fixing dates in the ecclesiastical calendars. With the gradual adoption of the Hindu numerals, during a period of much commercial activity, there was great interest in the science of calculating. Arithmetic was taught from the primary grade up in a purely mechanical way.

Influenced by the German masters of a half century earlier, Pestalozzi, at the beginning of the nineteenth cen-

tury, breathed new life into the teaching of arithmetic. Using *perception* as a basis he returned to object teaching, holding that an understanding of "figuring" should be preceded by a knowledge of the early number facts in connection with objects. Modern primary teaching shows the results of the good in Pestalozzi's teaching. The influence of Tillich was hardly less than that of Pestalozzi. He believed that a thorough understanding of all the operations in connection with the first ten numbers should form the basis for further work. Grube, in the nineteenth century, did much to systematize the teaching of the subject in the German schools.

The teaching of arithmetic in our own schools has been influenced largely by the English, the French, and the Germans, the latter having influenced us the most on the side of method.

Questions

1. Trace the origin of our present systems of writing numbers.
2. Distinguish between counting, notation, and numeration.
3. What is meant by the additive, subtractive, and multiplicative principles as applied to number symbols?
4. What is the principle of place value? What is the value of a symbol for zero in this connection?
5. Multiply 4236 by 384, using the first two of the Hindu methods mentioned above.
6. Divide 89,537 by 362, using the galley method.

CHAPTER II

THE REASONING INVOLVED IN ARITHMETIC

Logical types. The reasoning involved in the arithmetic of the school is in one respect like that employed in the ordinary thinking of every-day life. It is of such a nature that it is not easy to recognize in it the distinctive logical types. A similar statement may be made regarding most of the reasoning in algebra. Geometric reasoning, however, well illustrates, concretely, the types in formal logic. The logic of mathematics is essentially deductive, although inductive reasoning also plays an important part. A more elementary form of reasoning is that of reasoning by analogy, in which one concludes that if two things agree in a number of particulars, they agree in all. The method is not dependable and hardly deserves to be classed as a type of reasoning. All reasoning presupposes a formation of judgments, or accepted truths. In the earlier grades, where little reasoning is involved, much of the work consists in the stating of judgments, both by teacher and pupils. There are also the synthetic and analytic methods of proof, peculiar to mathematics, that provide avenues for the working out of the different types of reasoning. These are exemplified in arithmetic, but are more clearly defined in algebra and geometry. The teacher should be familiar with the forms of logic involved

in arithmetic, although she ought not to attempt to formulate these in any prescribed way for her classes.

Accepted truths. 1. *Definitions.* A first step toward reasoning in mathematics is that of getting together a body of principles or truths from which, through reasoning, other principles may be derived. Definitions are needed for purposes of identification and comparison. For example, the figure known as the square is studied. Its properties are noted, and the definition is framed that a square is that four-sided plane figure whose sides are all equal and whose angles are all equal. The pupil, when asked later to classify a certain figure, shows that it is a square by pointing out that its characteristics are identical with those expressed in the definition. Knowing, then, that the figure is a square, he may proceed with the principles that relate to the square. The definitions of decimal and non-decimal fractions enable one to classify any assigned fraction. Similar statements may be made with reference to the definitions of odd and even numbers and other definitions in arithmetic. A definition of subtraction, that the remainder added to the subtrahend gives as a sum the minuend, illustrates a different type of definition, one in which a plan of procedure is outlined. For instance, on being asked to get the difference between 8 and 5, we are expected to recall the number that must be added to 5 to give 8. In like manner the common definition of division directs how to obtain the answer. We are told, for example, that in order to divide 12 by 3 we must recall to our minds the number which if multiplied by 3 gives 12 as a product.

2. *Axioms and other assumptions.* Arithmetic makes use of a body of axioms common to both algebra and geometry. One of these, "If equals are added to equals, the sums are equal" is employed in the solution of the following simple problem: If \$4 less than John's money is \$8, how much money has John? Now \$4 added to an amount that is \$4 less than John's money must equal John's money. But \$8 represents \$4 less than the amount John has, from the statement of the problem. Hence \$4 added to \$8 must give the sum of money that John has. This is written algebraically, $x - 4 = 8$, from which we see that we add 4 to both sides of the equation to effect a solution. Axioms like the above are axioms of operation. Axioms of comparison are illustrated in "The whole is equal to the sum of all its parts," and "The whole is greater than any of its parts." The latter axiom is illustrated when we say that 8 is greater than 6. Another comparison axiom, "Things equal to the same things are equal to each other," plays an important part in mathematical reasoning, which reasoning is, after all, largely a matter of comparison. A simple illustration of the use of this axiom in arithmetic is: Since $\$2 + \$6 = \$8$ and $\$5 + \$3 = \$8$, we conclude that the sum of \$2 and \$6 is equal in value to the sum of \$5 and \$3. Another way of expressing this axiom is, that for anything its equal may be substituted. This elementary principle of substitution plays an important part in all mathematical reasoning. It is illustrated in an elementary way in ordinary addition, where it is required to add 2, 4, 5, and 7. The sum of 2 and 4 is 6. The substitution of 6 for the sum of 2 and 4

resolves the example into finding the sum of 6, 5, and 7, and so on.

Judgments. Judgments play an important part in forming the chain of relations that constitutes reasoning. One formulates a judgment in stating, without measuring, that a certain angle is acute or that two lines are of equal length: or that a group of 8 balls has been separated into a number of smaller groups. If an angle is found by measuring to be acute, a comparison has been set up, and we have no longer simple judgment. The axioms mentioned above are judgments of common agreement.

Reasoning by analogy. Reasoning by analogy, which generally enters unconsciously into the study of arithmetic, is untrustworthy. Two things may agree in a number of particulars and still not be identical. Circumstantial evidence should not have undue weight even in matters of reasoning. An illustration from arithmetic is not hard to find: $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$. By analogy the careless thinker may write $\sqrt{28} = \sqrt{25 + 3} = 5 + \sqrt{3}$ and think that he is applying the correct principle. The two processes do not agree in all particulars. For the first has the × sign under the symbol for square root and the second has the + sign. Again, by a careless use of the principle of analogy, pupils may write $\frac{\cancel{2} + 5}{\cancel{2}} = 5$, which is incorrect, whereas $\frac{\cancel{2} \times 5}{\cancel{2}} = 5$ is correct. This error commonly occurs in algebra, as in writing the incorrect statement $\frac{\cancel{a} + b\sqrt{3}}{\cancel{a}} = b\sqrt{3}$.

Deduction. The terms "reasoning" and "deductive reasoning" are commonly used synonymously, even in the field of mathematics, where inductive reasoning has considerable use. In deductive reasoning we proceed from the general to the particular. Some general law being established or assumed, deductive reasoning seeks to apply this law to particular cases. Suppose we have established the rule that the area of a rectangle is found by multiplying the length by the width, and, later, we are required to find the area of some particular figure. We classify the figure, say, as a rectangle; we recall the law for finding the area of a rectangle; we then make the required measurements, if not already given, and apply the law. The reasoning has been deductive. Deduction is employed throughout all of arithmetic wherever known principles are to be applied.

Genus and species. A principle in deductive reasoning is, that whatever is true of the genus is true of the species. For instance, we learn from a study of fractions objectively that a fraction is increased in size if the numerator is increased. This principle is recognized in dealing with both common and decimal fractions (the species) in computation. In the study of mensuration, the student learns that the area of a rectangle is found by multiplying the length by the width. The square, a special form of a rectangle, is treated similarly. The converse, that whatever is true of the species is true of the genus, is not true and should be guarded against. Some phases of the species agree with the genus, but not all. The sides of a square, for example, meet perpendicularly. So do the

sides of the more general figure, the rectangle. On the other hand, the diagonals of a square meet perpendicularly while the diagonals of other rectangles do not.

The syllogism. The process of passing from step to step in deduction is made possible through the use of the syllogism. The syllogism establishes a new truth from two properly chosen related truths. Truth *a* is a general proposition and may state some law or principle. It is called the major premise. Truth *b* tells something about both *a* and *c*. It is called the minor premise and contains no universal principle. The derived truth *c* tells something about both *a* and *b* and is called the conclusion. The following is an illustration from nature:

(*a*) All fish are cold-blooded.
(*b*) The salmon is a fish.
(*c*) Therefore the salmon is cold-blooded.

As a simple illustration from arithmetic we may formulate the syllogism:

(*a*) Every fraction whose numerator and denominator have a common factor may be reduced to lower terms.

(*b*) $\frac{8}{12}$ is a fraction whose numerator and denominator have a common factor.

(*c*) Therefore $\frac{8}{12}$ may be reduced to lower terms.

Some problems are solved by the use of two or more syllogisms. Consider the problem, "If 5 books cost $10, how much will 4 books cost?" In solving this problem by first finding the cost of one book four syllogisms are used, one for each of the logical steps and two for the mechanical solutions involved.

FIRST STEP

Syllogism (1)

(*a*) One book costs $\frac{1}{5}$ the cost of 5 books.
(*b*) The cost of 5 books is \$10.
(*c*) Therefore one book costs $\frac{1}{5}$ of \$10.

Syllogism (2). (To find $\frac{1}{5}$ of \$10)

(*a*) $\frac{1}{5}$ of \$10 is one of the five equal parts of \$10.
(*b*) One of the five equal parts of \$10 is \$2.
(*c*) Therefore $\frac{1}{5}$ of \$10 is \$2.

SECOND STEP

(*a*) 4 books cost 4 times the cost of one book.
The completion of this syllogism and the one that should follow is left to the reader.

By solving the above problem by means of the ratio idea the number of syllogisms is reduced.

Induction. Inductive reasoning leads from the particular to the general and is thus the opposite of deduction. It seeks to discover the general law that governs particular cases. The general law of gravitation was suggested to Sir Isaac Newton, we are told, from his observation of the falling apple. Many of the laws of science have been established inductively. Inductive reasoning is made reasonably secure through verification of the truth of the law in a great number of special cases. In this process of verification, deduction naturally enters. Examples of inductive reasoning in arithmetic are easily found: 3 balls and 5 balls are found, by counting, to equal 8 balls. Similarly 3 pins and 5 pins are found to equal 8 pins. Pupils readily conclude that 3 things of any kind and 5 things of any kind make 8 things all together. Again, let

it be required to find how to divide a whole number by a unit fraction: By means of a diagram, $4 \div \frac{1}{2}$ is found to equal 8. Similarly, $5 \div \frac{1}{3}$ is found to equal 15. The answer in each case is equal to the result obtained by multiplying the whole number by the denominator of the unit fraction. Hence $4 \div \frac{1}{2} = 4 \times 2$ and $5 \div \frac{1}{3} = 5 \times 3$, etc., the law being that just stated. It must be understood that proofs like the above are not complete inductive proofs. As has already been remarked, the deductive idea must be reckoned with to actually finish the argument. For instance, as soon as one shows that the method used in the above investigation is general, that it fits all cases in this class of numbers, the idea of deduction enters into the discussion, and the proof is hence made complete.

Synthesis and analysis. Synthesis may be defined as a process of putting together and analysis as a process of taking apart. Analyzing is examining and synthesizing is putting known elements together and getting new results. In analyzing we separate an element about which we are to learn something into known elements or elements that may become known. Much of the work of arithmetic consists of analysis and synthesis. Among the formal operations, the direct processes, — addition, multiplication, and involution, — are types of synthesis. In addition the known elements 2 and 3, for example, are put together and the result is a new element 5, — new in this association. The inverse processes, — subtraction, division, and evolution, — are types of analysis. In subtracting 2 from 5 we discover two of the elements that make up 5, the 2 that is taken away and the 3 that remains.

The term "analysis" is commonly associated with the explanation of problems in arithmetic. When pupils are told to analyze their work, they understand that the processes must be explained. But synthesis as well as analysis may enter into the explanation. For instance, in working the problem, "If 5 books cost $10, what is the cost of 4 books?" the first step involves analysis, in which the cost of one book is found. The second step involves synthesis, in which the cost of one book being known, the cost of four books is determined.

The terms "analysis" and "synthesis" as used traditionally in mathematics. Synthesis and analysis were not classed as methods of reasoning by the Greek logicians. As they were used by the early Greeks and as used to-day in the formal study of mathematics, these terms may be considered as pointing out two ways, either of which may be followed in the process of reasoning, but which give no further directions. Mathematical synthesis, in the sense here considered, proceeds from the known to the unknown. New truths are worked out from previously established principles. The theorems in our texts in geometry are demonstrated synthetically. For example, it may be required to prove that the area of a triangle equals one half the product of the base by the altitude. From the known principles that a triangle is equivalent to one half of a rectangle having a base and altitude equal respectively to those of the triangle, and that the area of a rectangle equals the product of its base and altitude, the required formula for the area of a triangle is readily deduced.

Mathematical analysis proceeds from the unknown to the

known. It seeks to discover the means by which a proof may be established. In geometry it may be used formally in both theorems and construction problems. The plan is to suppose the theorem true or the problem solved, and then seek to discover the lines of thought that connect it with known theorems and problems. This being done, the synthesis (proof) is given, in the case of theorems. In problems, the analysis is followed by the construction, and this by the synthesis. The analysis discovers and the synthesis proves.

Deduction in mathematical analysis and synthesis. The synthetic method, as the term is used in the present argument, is made possible through deductive reasoning, but mathematical synthesis and deduction are not identical, for mathematical analysis also uses deduction in arriving at its conclusions. For instance, it may be required to prove in geometry that the perpendicular is shorter than any oblique line drawn from a point to a line. By the method of mathematical analysis we examine the conditions that arise in case the desired theorem is true. In this step-by-step reasoning based on the temporary assumption that the perpendicular is shorter than any oblique line, we employ deduction through the use of the syllogism exactly as we should in case our assumption were a known fact.

Questions

1. Distinguish between induction and deduction; between analysis and synthesis.

2. Discuss the use of the terms "analysis" and "synthesis" in their broader logical significance and as used formally in geometry.

3. Express in syllogistic form the solution of the problem, "If 3 men do a piece of work in 10 days, how long will it take 1 man?"

CHAPTER III

PRELIMINARY STEPS IN ARITHMETIC

NUMBER

Fundamental aspect of number. Fundamentally speaking, number is independent of symbolism. The symbol 2 merely represents the number two. We get the conditions for the idea of number through the various senses, — sight, touch, hearing, smell, and taste. It comes to us, however, most frequently through the first of these. We may see a collection of objects or may hear successive taps of a bell. But in mere seeing or hearing we do not think number. Number has its origin in the counting process. It is a product of the mind and answers the question "how many," not "how much." We determine the "how many" in a series by counting the units that make up the series. Number is necessarily abstract.

Children like to count. It is a mistake, however, to conclude that because children count they understand number. The running through of a series of names, — one, two, three, etc., — may be different from counting a series of objects. Young children in attempting to count a series of objects are apt to run the series of names ahead of the things that their teachers think they are counting.

Number in measurement. Number may arise out of measurement. Thus we find how many times a unit is contained in a larger measurable quantity of the same kind.

This is generally illustrated by means of lines. The idea of ratio arises from a comparison of a unit with a collection of like units. The measuring and the ratio aspects of number are to be emphasized in teaching, but the serial aspect of number forms the basis for the art of calculating.

Cardinal and ordinal numbers. The cardinal numbers are merely the numbers used in the above description; that is, they define the "how many" in a group, and are written 1, 2, 3, 4, etc. If we wish to point out the position of numbers in a series, we write 1st, 2d, 3d, 4th, etc. These are called ordinal numbers.

Odd and even numbers. The odd numbers are the alternate numbers in the natural number series beginning with the first, — 1, 3, 5, 7, etc. The even numbers are the remaining alternate numbers, — 2, 4, 6, 8, etc.

Abstract and concrete numbers. The term "abstract number," strictly speaking, expresses a redundancy, for number is essentially abstract. The term is used in contrast to concrete number, where the numbered objects are named. We speak, for instance, of 5 dollars as a concrete number and of 5 as an abstract number.

The abstract idea of number should be considered in multiplication and division problems. In multiplication, the multiplier is abstract, the multiplicand and the product being concrete; thus, $16 multiplied by 2 equals $32. In division, the dividend is always concrete. If the divisor is concrete, the quotient is abstract; for instance, $8 divided by $2 gives 4 as a quotient. If the divisor is abstract, the quotient is concrete; thus, $8 divided by 2 (that is, divided into two equal parts) gives $4 as a quotient.

SERIAL COUNTING

Basis for computation. Counting is the basis of all computation in arithmetic. Out of it grow, in turn, addition, multiplication, and involution. The inverse processes, — subtraction, division, and evolution, — follow from these. Since counting is of prime importance, pupils should early be made familiar with the number series. Counting by means of objects, while of value in connection with the numbers up to ten, or perhaps a few beyond ten, is a hindrance rather than an aid in bringing this about.

Number series and number spaces.[1] Children count first by ones without the aid of number symbols. They may be able to count also by twos, tens, etc., but the aid given through the sense of sight by the use of the numerals should be recognized by the teacher in the early work. Whatever the device used, the pupil should know the relative positions of numbers in the natural series. This knowledge can be readily secured by considering the decades, 1–10, 10–20, 20–30, etc. Ask questions as follows:

Count by tens, beginning with 10; also beginning with 1, 2, 3, 4, etc.
What decade follows the decade 10–20, etc.? precedes 60–70, etc.?
Count by fives, beginning with 5.
What is the first number after 4, 24, 34, etc.? after 52, 62, etc.? after 39, 49, etc.?
What is the first number before 7, 17, 27, etc.? before 39, 49, etc.?
What is the second number after 5, 15, 25, 55, etc.? after 47, 73, etc.? after 8, 18, 28, etc.? after 9, 19, 29, 39, etc.?
What is the second number before 7, 17, 27, etc.? before 28, 54, 83, etc.? before 10, 20, 30, etc.? before 11, 21, 31, 41, etc.?
What is the first number after 28 that ends in 1, etc.?

[1] See topic "Place of Number in the Series," pp. 267, 269, 271–272.

Make use of the following number table, which should be on the board.

Number Table:

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

Starting with any number in the first column, count by tens by reading the table horizontally. Erase all the even numbers and have the class rewrite them. Do the same with the odd numbers. If possible, have the pupils name the numbers orally as they are written in so as to fix the series of even and odd numbers. Have the pupils build the table in a variety of ways. One way is to write in the series of even numbers and then the series of odd numbers. Again write in the row 10, 20, 30, etc. Then 5, 15, 25, etc., and finally the other rows, beginning with the first. As soon as the class can count fairly well serially by fours, eights, threes, sixes, etc., let them begin building the table according to these series and then complete it by counting within the columns or within the rows. In this way errors may be detected. A good device in teaching any series is to insert circles or squares in place of the numbers of any particular series and have the class fill in the series. Another good plan is to print in heavy type the series to be learned. Since serial counting is the basis for learning the multiplication tables, its importance should not be underestimated.

There should be drills in reading and writing numbers from 1 to 100, in which the numbers follow one another in order but not in groups of tens as in the above table.[1] This may be necessary so that the pupils may feel that 11, 21, etc., respectively follow *next to* 10, 20, etc. In other words, the continuity of the natural number series should be preserved in the mind of the pupil.

[1] See suggestion for number chart, p. 267.

READING AND WRITING NUMBERS

When to begin notation and numeration. Children generally count and recognize simple number groups before they learn number symbols, but when real instruction in arithmetic is begun, the pupil should learn, for example, the symbol 6 when he names the number represented by six dots. The written word "six" may be taught at this time. Thus the pupil sees the number six, hears the word six, sees the symbol 6, and sees the written word six.

● ●
● ● six 6
● ●

Pupils write numbers as high as 100, and perhaps beyond, before there is need of emphasizing the idea of numeration. But whenever numbers as high as 1000 are written, the idea of place value, or numeration, should be taught.

The formal side of reading and writing numbers. In the case of entrance pupils who do not know how to read or write even the simplest numbers, it is advisable to instruct in a very systematic fashion. After making sure that the pupils have some facility in counting, teach them the reading and writing of the numbers in the regular order, 1, 2, 3, 4, etc. Teach a new number only after all the numbers previously taught can be readily read, both when written serially or at random by the teacher.

In teaching the reading and writing of the larger numbers, care should be used in the grading, even when the same number of figures is used in the several numbers. To illustrate, in each of the following sets of numbers the

more difficult are approximately in order: 800; 820; 807; 3000; 3400; 3450; 3407; 3057; 3005.

SENSE TRAINING

Basis for application. The term "sense training," as here used in relation to mathematics, has to do with giving the pupils quantitative experience and with giving them the basis for appreciating the applications of arithmetic. On the side of number they learn concretely, from objects, the significance of the "how many," and on the geometric side, the "how much." The ideas of greater than, less than, and equal to, are brought out, both in the numerical and geometric aspects. The notions of distance, direction, and limited portions of time and space are put before them concretely. Care should be taken that children get correct language forms in such work. Sense training in the form of measurement extends throughout the elementary course and includes work in denominate numbers and in mensuration.

Nature of drills. 1. *Counting and number grouping.* It is necessary to associate the counting of the first ten numbers with objects, in order that the pupils may, when occasion demands, apply arithmetic to their needs. This drill should be accompanied by exercises in recognizing groups of numbers, either when irregularly arranged in the case of the first four or five numbers, or when arranged in special forms, as on dominoes. Other senses than the sense of sight should be appealed to. Counting the taps of a bell adds to an understanding of number through the sense of hearing.

2. *Recognizing geometric forms.* Since arithmetic is ap-

plied extensively in determining space relations, the names and properties of the common geometric forms should be taught early and as far as possible with reference to the mathematical purpose for which they are to be used. The familiar plane figures, — the circle, semicircle, square, rectangle, and triangle, — may be studied through drawings made first by the teacher and then by the class; the familiar solids, — the sphere, hemisphere, cube, cylinder, and cone, — from the objects themselves. Among other things the notions of line, angle, right angle, vertex, surface, and solid should be brought out.

3. *Notions of greater than, less than, and equality.* From a comparison of the familiar plane figures and from geometric solids lead the pupils to tell which is the greater or less of two objects, and which is the greatest or least in a group. Next identify objects that are equal in size. The same drills may be used in the comparison of weights.

4. *Comparison by means of measurement.* Exact comparison with respect to relative magnitude must be secured through measurement. The ratio aspect of number is here recognized. Pupils compare the 1 inch with the 2 inch, the 2 inch with the 4 inch, the 3 inch with the 6 inch and the 12 inch, etc., the foot with the yard. Squares and rectangles divided into smaller unit squares are compared. Similarly, cubes and rectangular parallelepipeds may be divided into smaller cubes as units. In such work the class is made familiar with the terms "times" and "part of." The pupils learn that line a is 4 times line b; solid e is $\frac{1}{8}$ of solid c, and so on. The foundation is here laid for ratio and fractions.

Work like the above, together with drills involving comparisons between the five-cent piece, the dime, and the dollar and between the pint, the quart, and the gallon, gives the foundation for later work in denominate numbers.

Paper folding. Paper folding provides an interesting and instructive means of teaching children early number relations. Colored paper squares, such as are used in the kindergarten, should be secured. The following brief directions indicate the method of procedure:

Fold once, the bottom side fitting the top. Unfold. Ask questions bringing out the $\frac{1}{2}$ and 2 times.

Fold as before and then from right to left. Unfold. Bring out the ideas of 4 times and $\frac{1}{4}$. Also the 2 times and $\frac{1}{2}$ with reference to the divided $\frac{1}{2}$ of the whole.

Refold and continue folding to bring out other fractional relations.

Fold a fresh square from the bottom into three equal horizontal strips. Open and fold left side on right side. The open paper will give a diagram for studying 3 times and $\frac{1}{3}$, 2 times and $\frac{1}{2}$, 6 times and $\frac{1}{6}$, etc. Other relations may be brought out by folding the paper in various ways.

EARLY NUMBER FACTS

The fundamental operations. The four fundamental operations in arithmetic are addition, subtraction, multiplication, and division, the latter considered in its dual aspect,—division by measuring and division by partitioning. Before the pupil is ready for the ordinary written drill work involving these operations, he should study them in connection with the numbers 1 to 10, or 1 to 20, and through the use of objects.

Objects and memory work. Teachers are not agreed as to the method of beginning the abstract work in number — whether to use the objective work as a necessary preliminary for the memorizing of the number fact, that is, to have the children learn, for example, from the objects that 9 and 4 are 13 and memorize the result; or to have them merely learn that 9 and 4 are 13 and not relate the fact to objects or counting. From the standpoint of memory and mechanical efficiency, the child is not aided by any objective presentation. He is probably more hindered. Still, objective work is essential from practical considerations. The following plan has much in its favor: Teach objectively the early number facts, not requiring the pupils to memorize results, although they will naturally learn some by heart. This work is done during the first two years and while the children are still counting, reading, and writing numbers, and studying comparative magnitudes. Beginning in the High Second or Low Third, follow this work with memorizing the addition combinations and other number facts that enter into computation. In the following discussion, which is topically arranged, it has not been possible to outline the work in the sequence of class work. The reader is referred to the detailed work of the first year and a half on pp. 264–277.

The complementary contents. In speaking of the complementary contents of the early numbers, reference is here made to the addition and subtraction facts. When studying the number 6, for example, the pupils find that $5+1=6$, $1+5=6$, $6-1=5$, $6-5=1$; $4+2=6$, etc. The numbers are usually studied in order from 1 to

10. Objects find a place here. Before any number is discarded for the next, teach the subtraction facts, using the "take away" and the "making up" ideas. We may say 2 from 7 leaves five, or we may ask the question, what number added to 2 gives 7. The latter form suggests a previous experience (in addition), and hence has the greater claim in the choice of methods in written subtraction.

The measure contents. The multiplication and division facts are associated in studying the measure contents of numbers. After teaching the complementary contents of the numbers 1–10, teach the measure contents. It may be convenient to teach the latter in close connection with the former, but in any case avoid the errors of the Grube Method in exhausting all the number facts in connection with one number before taking up the next.

Objects are especially serviceable in teaching the measure contents of the early numbers. Here the pupil learns, for example, that 3 groups of blocks, 2 blocks in each group, make 6 blocks (multiplication).

Also if 6 blocks are divided into groups with 2 in each group, 3 groups are formed (division by measuring or merely division).

Also if 6 blocks are divided into three groups with the same number in each group, 2 blocks are in each group (division by partitioning, or partition).

In other words, we have 3×2 blocks $= 6$ blocks (multiplication); 6 blocks $\div$ 2 blocks $= 3$ (division); and 6 blocks $\div 3 = 2$ blocks (partition). We generally explain the latter by saying $\frac{1}{3}$ of 6 blocks is 2 blocks.

Three so-called story problems, of which the above are

solutions, are: (1) Each of 3 little girls has 2 blocks. How many have they in all (multiplication)? (2) 6 blocks were divided among some little girls so that each received 2 blocks. How many little girls were there (division)? (3) Three little girls wished to divide equally among themselves 6 blocks. How many blocks would each little girl receive (partition)? (Compare this discussion with that on p. 30 with respect to abstract and concrete numbers.)

In teaching the measure contents, make use of the comparison of magnitudes as explained in the preceding topic, but seek variety by using the colored sticks and other discrete magnitudes. Do not take numbers necessarily in order. The even numbers are more simple than the odd. The number 7 cannot be compared in the above sense with any number before 14 is reached, but 5 can be compared with 10. Relate the problems here involved to the pupil's experiences. Give problems involving the use of the five-cent piece and the dime; the foot and the yard; the pint, the quart, and the gallon; the foot and twelve inches; etc.

Questions

1. What are the complementary and the measure contents of the number 8?

2. Which of the following examples is partition and which division?

(*a*) If one hat can be bought for $4, how many hats can be bought for $8?

(*b*) Three boys have equal shares in a sled that cost $6. What is the value of each boy's share?

(*c*) If 3 pairs of shoes cost $15, what is the cost of one pair?

(*d*) How many times must a three-quart measure be filled in order to fill with it a twelve-gallon vessel?

CHAPTER IV

THE PRINCIPAL OPERATIONS IN ARITHMETIC

NOTATION AND NUMERATION[1]

Fundamental principles. In notation we are concerned with symbolizing numbers, and in numeration with the idea of place value. If all arithmetic were mental, we could possibly do without a system of notation, but a system of numeration would be indispensable. It would be possible, however, to have a crude arithmetic without either notation or numeration. This would necessarily be mental, and there would not be any grouping by tens or other numbers. It would be possible to have an arithmetic employing notation without numeration. This would necessitate a separate symbol for each of our numbers in order. In our present (Arabic) system of notation we employ nine different symbols for the first nine numbers. Since our system of numeration has ten for a base, the first number after nine is written 10, in which no new symbol is employed except the 0. (See Questions 1 and 9 below.) If our base were seven instead of ten, the first number after six would be written 10. In our present system numbers beyond ten do not make use of new symbols but employ new combinations of the first

[1] For suggestions concerning the reading and counting of numbers in the earlier grades, see p. 33.

nine symbols and 0. Thus we write 11, 201, 3289, etc. If no base like ten were employed, it would logically follow that there would be as many different number symbols as there are numbers. We thus see that in our common system of writing numbers, notation and numeration are intimately associated.

Points to emphasize. The ability to read and write numbers correctly implies an understanding of the following points:

1. The names of the different orders, — units, tens, hundreds, etc.
2. The relations between these orders.

   In 326, for example, one of the 10's equals 10 of the 1's (units); one of the 100's equals 10 of the 10's; one of the 100's equals 100 of the 1's.

3. The highest digit possible in any order is 9.
4. The names of the different periods, — units, thousands, millions, etc.
5. The units' place for any period is in the right-hand order of that period.

   Consider, for example, the thousands' period in 376,425,819. There are 425 thousands. One thousand is the unit for this period, and there are 425 of them. The above number also comprises 376 millions. One million is the unit for this period, and there are 376 of them. The entire number written shows, of course, the number of fundamental units represented; that is, there are 376,425,819 of them.

   The essential ideas outlined above can best be brought out by colored sticks or toothpicks, placed in bundles of 10, 100, etc. Numbers should be built, then read, and then written. Other numbers should be written by the teacher, then read and built by the pupils. A box with compartments to hold the different

sized bundles of sticks will be found of advantage, or lines may be drawn vertically on a table to provide a place for the different orders. The teacher may well appeal to the visualizing power of the class and thus be able to omit much detailed work with the objects. A reference to money helps to make matters clear. For example, $324 may be composed of 4 dollar bills, 2 ten-dollar bills, and 3 hundred-dollar bills. There should be much board work in reading and writing numbers and naming the orders and periods. A poor understanding of notation and numeration of whole numbers is apt to result in a poor understanding of decimals. The fundamental difficulty is that the pupils have not a clear idea of place value. The common method of using a comma to separate the different periods is to be recommended.

6. Avoid the use of "and" in reading whole numbers.

Questions

The following are intended as exercises in arithmetic for the reader. Some of the higher arithmetics give an explanation of work like that which follows. Brooks's *Philosophy of Arithmetic* goes into it quite fully.

1. Add
$$\begin{array}{r}324\\132\\\underline{623}\end{array}$$
, the base being eight. Note that 10 here represents eight, not ten. In the first number 324, the 2 represents 2 bundles or groups with eight in a bundle; the 3, 3 bundles with 8×8 in a bundle. It will be profitable for the inexperienced teacher to build numbers out of sticks, using eight or any other base than ten. Such work, of course, is not intended for pupils. The answer to the above example is 1301. Note that this is not read one thousand three hundred one.

2. Subtract
$$\begin{array}{r}523\\\underline{134}\end{array}$$
, where the base is six. *Ans.* 345.

3. Solve (1) and (2) using bases nine, ten, seven.

4. Why could not 623 have five for a base?

5. Try to write eleven thousand eleven hundred eleven as one number. What relation does this bear to the preceding question?

6. Write as decimals $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{8}$.

7. Write the same as duodecimals (base twelve).

Ans. $\frac{1}{2} = .6$ $\frac{1}{4} = .3$
$\frac{1}{3} = .4$ $\frac{1}{8} = .16$

8. Explain the difference between the English and the French systems of numeration. Consult the dictionary for the answer.

9. The following table shows how to write numbers from one to twelve with bases from two to twelve:

BASES	1	2	3	4	5	6	7	8	9	10	11	12
Two . .	1	10	11	100	101	110	111	1000	1001	1010	1011	1100
Three .	1	2	10	11	12	20	21	22	100	101	102	110
Four .	1	2	3	10	11	12	13	20	21	22	23	30
Five . .	1	2	3	4	10	11	12	13	14	20	21	22
Six . .	1	2	3	4	5	10	11	12	13	14	15	20
Seven .	1	2	3	4	5	6	10	11	12	13	14	15
Eight .	1	2	3	4	5	6	7	10	11	12	13	14
Nine .	1	2	3	4	5	6	7	8	10	11	12	13
Ten . .	1	2	3	4	5	6	7	8	9	10	11	12
Eleven	1	2	3	4	5	6	7	8	9	ϕ	10	11
Twelve	1	2	3	4	5	6	7	8	9	ϕ	π	10

Note that when the base is greater than ten, single symbols are used for all numbers less than the base.

ADDITION

The forty-five combinations. There are forty-five different combinations resulting from adding the numbers from 1 to 9 inclusive, taking them two at a time and without repetitions. They are:

$\frac{10!}{2!\,8!} = \frac{10 \times 9}{2} = 45$

$\frac{9!}{2!\,7!} = 36$

1 1	1 2	1 3	1 4	1 5	1 6	1 7	1 8	1 9
2 2	2 3	2 4	2 5	2 6	2 7	2 8	2 9	
3 3	3 4	3 5	3 6	3 7	3 8	3 9		
4 4	4 5	4 6	4 7	4 8	4 9			
5 5	5 6	5 7	5 8	5 9				
6 6	6 7	6 8	6 9					
7 7	7 8	7 9						
8 8	8 9							
9 9								

We speak of the thirty-six combinations when the first nine in the above table are omitted. Some of these combinations are memorized in connection with the learning of some of the other early number facts. It is not desirable that teachers should teach the combinations in the order given in the table.

The learning of the combinations is necessary for the study of column addition, and hence they should be thoroughly memorized. The pupil should know that 4 and 5 are 9 just as positively as he knows that b–o–y spells *boy*. The combinations whose sums are not over 10 are

usually taught first.[1] The pupil is familiar with reading and writing numbers at least as far as 100, hence when he learns that 5 and 2 are 7, he can pass readily to 25 and 2 are 27, 35 and 2 are 37, etc. The combination 15 and 2 are 17 is apt to be harder than the others mentioned on account of the sound of the "teen." Drill on the fact that 2 added to a number ending with a 5 always gives a number ending with a 7. Referring to the table on p. 44, we observe that there are twenty-five combinations whose sums are less than 11. There are ten decades, 1–10, 10–20, etc. up to 100. Hence when the pupil learns the twenty-five combinations of numbers whose sums are less than 11, he can easily be taught to give 225 others in the number space 10–100.

The remaining twenty of the forty-five combinations have as their sums numbers ranging from 11 to 18. The numbers added are in the first number space, 1–10, the sums being in the second number space, 10–20. While the pupil should learn reflexively that 9 and 4 are 13, he may find it helpful to have some basis upon which to correct a wrong answer. If, for example, he is not sure of $9 + 6$, he may be helped by recalling that $9 + 6$ lies between $9 + 7$ and $9 + 5$. Again, it being known that $8 + 8 = 16$ and $7 + 7 = 14$, it is easy to remember that $8 + 7 = 15$ or $7 + 8 = 15$. It may be helpful to emphasize that the sum of two even numbers and also of two odd numbers always gives an even number. Also that the sum of an odd and an even number always gives an odd number. Such correctives as these are of doubtful value while the teacher is

[1] See p. 47 for another plan.

seeking to have answers memorized, but may prove helpful after the work is well in hand. Pupils should not use counting as a crutch for getting the answer to any combination. For example, they should be discouraged from finding the sum of 5 and 3 by counting 3 beyond 5. When columns are to be added later, the counting habit should not stand in the way of accurate and rapid addition.

After the pupils have memorized, for example, $9 + 4 = 13$, they can easily learn $19 + 4 = 23$, $29 + 4 = 33$, etc. They thus learn that adding 4 to any number ending with 9 gives the next number that ends with 3, and similarly with other combinations.

Not only should pupils know the combinations by heart, but they should be able to give the complementary contents of all the numbers up to 20. They should be asked, for example, to give all the combinations that make 15: 15 equals $8 + 7$, $7 + 8$, $9 + 6$, $6 + 9$, $10 + 5$, $5 + 10$, $11 + 4$, etc. The subtraction facts should be learned in close connection with the addition facts.

The law of commutation should be remembered by the teacher. To illustrate: when the pupil learns $9 + 4 = 13$, he is immediately taught $4 + 9 = 13$; that is, $9 + 4 = 4 + 9$.

The essential thing is that the combinations be thoroughly memorized. Primary arithmetics usually give devices that aid in this memory work. Cards with the combinations on one side and the answers on the reverse side may prove helpful. The teacher flashes the card before the class for an instant and a pupil is expected to give the answer. It is always well to give the class some incentive for doing good work, such as writing the names

on the board of all those giving correct answers. The teacher should also give quick mental drills by the question and answer method.

First steps in column addition. The learning of the forty-five combinations forms the immediate basis for column addition. Pupils should learn the sums in examples like $\begin{array}{r}13\\+\underline{\;2}\end{array}$ $\begin{array}{r}24\\+\underline{\;5}\end{array}$ $\begin{array}{r}7\\+\underline{15}\end{array}$ before long single columns are added. The common practice is to have the forty-five combinations and examples like the above thoroughly memorized before beginning column addition. Pupils usually have difficulty, however, in applying the combinations while adding the column. One of our recent texts proposes a plan for solving this difficulty.

The main features of this plan for teaching column addition are:

Step A. Select for special drill a group of five combinations in which the right-hand digit of the sum in the first is the lower number in the second, the right-hand digit of the sum in the second is the lower number in the third, etc.

$\begin{array}{ccccc}3 & 4 & 3 & 2 & 6\\ \underline{2} & \underline{5} & \underline{9} & \underline{2} & \underline{4}\end{array}$ Also teach the reverses.

Step B. Think of 10, 20, 30, etc. added to each of the lower numbers in the combinations of Step A and add the top numbers.

Step C. Build columns that depend upon the combinations learned in Step A, using them from left to right. The columns are built and added from the bottom up. For instance, the first of these two columns is made up from the first two combinations in Step A and the second from the first three. A simple way to make up the column is to write the first combination and then write in turn above it the top numbers in the remaining combinations. Other groups of combinations are applied in similar fashion. In due time pupils should add in both directions.

$\begin{array}{cc} & 3\\ 4 & 4\\ 3 & 3\\ \underline{2} & \underline{2}\end{array}$

Great emphasis should be laid on column addition since it is the basis of all the mechanical work that follows in the higher grades. One who adds correctly is apt to be accurate in all his figuring. Drills in addition should continue throughout the grades until the pupils are accurate and relatively rapid.

Suggestions in teaching addition. 1. Give mental exercises requiring the sum $3 + 4 + 10 + 5 +$ etc. These may be varied later by using also subtraction, multiplication, and division. Thus the teacher says 6, multiply by 3, add 2, divide by 4, subtract 1, add 11. The commas used have the effect of parentheses. Pupils may record their answers for correction.

	+3
6	9
3	etc.
2	
4	
5	
7	
9	
8	
1	

Another device is to have the pupils write a column as in the illustration, place the number to be added at the top above the horizontal line at the top, and write answers at the right of the vertical line. A similar plan may be followed later, in subtraction, multiplication, or division, by placing the subtrahend, the multiplier, or divisor at the top.

2. Add mentally numbers like 45 and 35, 39 and 52. Observe the principle of regrouping. To add the first of these the pupil may add 40 and 30 and then 10; the second, 40 and 52 and subtract 1. Add as one would when paying for articles bought in a store. In other words seek to add the highest orders first. An observance of this principle enables one with practice to add two or more columns at a time in written addition.

3. Lead the pupils to observe the principle of grouping

in adding the separate columns. We can frequently group by 10's. In adding this column we say, 10, 20, 33, leaving the 8 and 5 to the last. There are certain dangers in doing this, but it is frequently of service. In adding the above the average pupil can at least say 9, 15, 25, 33.

$$\begin{array}{r} 8 \\ 3 \\ 7 \\ 6 \\ 5 \\ \underline{4} \end{array}$$

4. In column addition do not write the numbers "carried" either at the top or the bottom of the columns. If, however, there are several long columns, in seat work the pupils may write the numbers "carried" on a slip of paper placed beneath the columns; in board work the extra figures may be placed below the sum or to one side and later erased.

Checks. As a check in addition add the columns in both directions. In the upper grades checking by casting out the 9's or 11's may be used. The method by casting out the 9's depends upon the principle that the remainder resulting from dividing any number by 9 may be found by adding the digits of the number and dividing the sum by 9. The resulting remainder is the required remainder.

For example, in 2465, by adding the digits we obtain 17. Dividing 17 by 9 gives a remainder of 8, which is the remainder that we should obtain if we divided 2465 by 9. The nines may be subtracted as we add the digits. To illustrate, in 2465 the 5 and the 4 make 9 and the sum of the 6 and the 2 gives 8, the remainder. The remainder 8 is called the excess.

In checking the addition of any column of numbers, write the excesses of the addends to the right in a column. Add the excesses and cast out the nines of the sum. This

result should equal the excess in the sum of the numbers added in the example. If not, there has been an error.

Ex.

	2467	1	
	5284	1	
	9623	2	
	4246	7	
	5684	5	
	7341	6	
4	34645	22	4

Checking by casting out the 9's or 11's does not always reveal the error. It is obvious that these methods of checking are only of value to the average person in case long columns are added. Full explanations of these methods are given in the higher arithmetics.

Questions

1. Following the plan of teaching column addition described on p. 47, what combinations should be given in Step A to prepare for adding the column?

$$\begin{array}{r} 6 \\ 4 \\ 3 \\ \underline{5} \end{array}$$

2. Add 231; 628; 926; 415, and check by casting out the 9's.

3. Articles worth \$.85, \$.25, and \$.40, respectively, were purchased. Find the total cost mentally. SUGGESTION: First add \$.85 and \$.25. \$.15 added to \$.85 makes \$1.00. \$.10 added to \$1.00 makes \$1.10. Hence the sum of \$.85 and \$.25 is \$1.10. \$1.10 + \$.40 = \$1.50.

Find the sum of 85, 25, and 40; 35, 45, 20, and 80.

Find the sum of 39 and 62. SUGGESTION: Add 40 and 62 and subtract 1. Find the sum of 59 and 31.

4. An interesting property of the number 9 is found in the following diversion. The teacher asks any one to write the first two numbers, as in the column here given. The teacher then writes in the third number, some one else the fourth, the teacher the fifth, and so on alter-

nately. The answer can be written without adding by observing the following rule: Annex to the left of the top number the figure that represents the number of numbers the teacher has written down (3, in this case) and subtract from the top number the number represented by this same figure (3, in this case). The answer here is 324,672. The trick is discovered by examining the figures the teacher chooses in writing down his numbers. Thus, 16,485 + 83,514 = 99,999; 35,467 + 64,532 = 99,999; 98,204 + 1795 = 99,999. The sum of the numbers to be added to the first number in the column equals 3 × 100,000 less 3.

$$\begin{array}{r} 24675 \\ 16485 \\ 83514 \\ 35467 \\ 64532 \\ 98204 \\ \underline{1795} \end{array}$$

The following game also illustrates the peculiar properties of the number 9. Ask any number of pupils to write down any number they wish. Ask them to add the digits and subtract the sum from the original number, and then cross out any digit in the remainder that is not a 0 or a 9. Ask for the sum of the remaining digits. The teacher is able to tell, in each case, the digit crossed out. This is told by the following principle: The number crossed out is the difference between the sum of the remaining digits and the next following multiple of 9. Ex.: Write down 257,649. The sum of the digits is 33. The difference between 257,649 and 33 is 257,616. Cross out the 7. The sum of the remaining digits is 20. The multiple of 9 following 20 is 27. The difference between 27 and 20 is 7, the number crossed out.

SUBTRACTION

Fundamental ideas involved. Before the pupil subtracts in the ordinary written form he is familiar with three fundamental ideas of subtraction, illustrated in the three following ways of stating the example:

(*a*) 7 less 3 is what? — the take-away idea.

(*b*) What number added to 3 gives 7? — the making-up idea.

(*c*) 7 is how many more than 3? — the difference-between idea.

According to the first of these ideas one may picture 7 objects and then 3 taken away; according to the second,

3 objects and other objects added to these until the total equals 7; according to the third, 7 objects in one group and 3 objects in another group. Notice that in addition we may first picture 3 objects in one group and 4 objects in another group. In the preceding types we have given the forms of subtraction problems. The method of performing written subtraction is another matter.

Written subtraction. There are three common methods in written subtraction. It is perhaps of no great consequence which method one teaches, but the Austrian, or addition, method has much in its favor. It relates very closely to the experience gained while learning the addition combinations and corresponds with the method of counting change in business transactions. This method is illustrated in the following examples:

In subtracting 23 from 57 we say 3 and 4 are 7, writing down the 4; 2 and 3 are 5, writing down the 3. $\begin{array}{r}57\\ \underline{23}\end{array}$ In subtracting 29 from 57 $\begin{array}{r}57\\ \underline{29}\end{array}$ we see that 9 is larger than 7 and we must answer the question, 9 and how many are 17? Hence we say 9 and 8 are 17, writing down the 8. Since our sum is 17, there is 1 to carry. Add the 1 to the 2, giving 3. Then we say 3 and 2 are 5, writing down the 2. The pupil should remember the two steps, thinking the 17 and adding 1 more to the 2 in the subtrahend. This form of statement may be preferred by some: 9 to make 17, 8. 3 to make 5, 2. It preserves the idea of addition, but differs from the language used in addition.

The fundamental idea in the above method is that the minuend is the sum of the subtrahend and the remainder, or difference. It may be helpful to let the pupils add two numbers like 29 and 28, the 29 being written first and the sum written above instead of beneath. Then let them erase the 28 and they have the above example in subtraction.

The so-called borrowing method:

In $\begin{array}{r} 57 \\ -\underline{29} \end{array}$ we cannot subtract 9 from 7 so we take 1 ten from the 5 tens, change it to 10 ones, and add these to the 7 ones in units' place, making 17; 9 from 17, 8. Since 1 ten was taken away from the 5 tens, we must say 2 from 4 in the second column. Bundles of splints may be used at first, but this should not be necessary. The pupil may imagine a 1 to the left of the 7 and may at the proper time think of a 4 in place of the 5. There should be no scratching out and writing in of numbers.

An add-to-the-subtrahend method:

In $\begin{array}{r} 57 \\ -\underline{29} \end{array}$ we cannot subtract 9 from 7 so we think 9 from 17, thereby adding 10 to the minuend before subtracting; 9 from 17, 8. Having added 10 ones to the minuend, we add 1 ten to the subtrahend to counteract this. 1 ten added to 2 tens gives 3 tens. Then we say 3 from 5, 2.

The teacher will, of course, teach but one method of subtraction to a beginning class. The method being chosen, she should shape the early drills accordingly. If the Austrian method is adopted, the pupils will from the first say in written work 3 and 4 are 7 (or 3 to make 7, 4) when getting the difference between 7 and 3.

Suggestions in teaching subtraction. 1. Use devices similar to those given in Suggestion (1) for addition on p. 48.

2. Teach "making change," using toy money. In making change, use the common business method in which the making-up, or addition, idea is involved. This argues for teaching the Austrian method of written subtraction.

3. Use the same idea in mental subtraction. For example, in subtracting 29 from 50, say 1 added to twenty-nine makes thirty, and 20 more makes fifty. Hence the

difference is 21. In practice we merely say 1, 20, 21. This is exactly what we say in counting our change when buying an article for 29 cents, having given the clerk 50 cents.

4. Check written subtraction by adding the subtrahend and the remainder. The result should equal the minuend.

Questions

1. Why is it not strictly correct to use the word "borrow" in the second method of subtraction given above?

2. In what three forms should subtraction problems be given, especially in early work?

3. An article cost $13.85. The customer gave the clerk $20. How should the clerk count out the change? What change is returned?

4. From 20 subtract 13.85.

5. From 2,000 subtract 1,385.

MULTIPLICATION

Teaching the tables. In the early work when the pupil learns, for example, $2 + 2 + 2 = 6$, he is taught to say three 2's are six. This inaugurates the learning of the tables, but the drills in serial counting give the broad basis for their mastery. If it is desired to teach the table of 4's, the pupils first count 4, 8, 12, 16, and then say one 4 is 4, two 4's are 8, three 4's are 12, four 4's are 16. The following form of writing the series will probably aid in adding meaning to the expressions, one 4 is 4, two 4's are 8, etc.:

4	4	4	4	4	etc.
	4	4	4	4	
		4	4	4	
			4	4	
				4	
4	8	12	16	20	

A generation past it was a common practice to have pupils first memorize all the tables and then apply them in examples. The method in common use to-day is to apply each table after it is learned in both multiplication and division examples. One recent text plans to have products memorized without reference to any serial order. The plan as there worked out prevents any possible waste of time in learning a table and then learning it anew when products are asked for at random. Whatever the text followed, it will probably be found that some pupils will be greatly helped by "saying" the tables or parts of tables in the old-fashioned way, one 7 is 7, two 7's are 14, etc. Most texts give devices for learning the products at random, the most common being the circle device, in which numbers written on the circumference are to be multiplied by the number at the center. Variety is secured by changing the multiplier. Sides may be chosen and the class "spell down" as in a spelling match.

After the tables are well in hand, a table like the following may be built by the pupils:

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15 etc.									
6											
7											
8											
9											
10											
11											
12	24	36	48	60	72 etc.						

If the left-hand column is taken as the column of multipliers and the top row as the row of multiplicands, the products are written in the successive rows. If the multipliers and multiplicands are interchanged, the products are found in the successive columns, beginning with the second column on the left. Note that one may count serially in the columns or in the rows. In later work this table may be used to study ratio. On examining the second and third rows it is seen that the ratio of 2 to 3 equals the ratio of 4 to 6, 6 to 9, 8 to 12, etc. Also the ratio of 3 to 5 equals the ratio of 6 to 10, 9 to 15, etc. One of the diagonals gives the series of numbers that are perfect squares.

Applying the tables. As soon as each table is learned, apply it in written multiplication, using first a single digit as a multiplier. The pupil may safely multiply by 23 before he has mastered the table of 4's. Short division should be taught after a few of the tables are learned. When the pupil can multiply by 5, let him divide by 5.

The use of 62, 63, 64 as multiplicands should precede those like 26, 36, 46, where 2, for example, is the multiplier, thereby avoiding the carrying idea in the first examples.

Note that $\begin{array}{r} 46 \\ \underline{\times 2} \end{array}$ is but an abbreviation for the sum $\begin{array}{r} 46 \\ \underline{46} \end{array}$.

In using multipliers of two digits there should not be any tedious reference to the various orders. In products like 549 multiplied by 32, the class must be sure of the following steps:

(*a*) 549 must first be multiplied by 2 and then by 3.

(*b*) The first digit of each partial product must be placed directly beneath the digit being used in the multiplier.

(*c*) The partial products must be added.

For those teachers who prefer a fuller development, the following may be at first required:

549	
32	
1098	2 × 549
16470	30 × 549
17568	

Checks. In the later grades, teach checking multiplication by casting out the 9's. The method is illustrated in the accompanying example.

	6284	2	
	547	7	
	43988	14	5
	25136		
	31420		
5	3437348		

Cast out the 9's in both the multiplicand and the multiplier. Multiply the excesses. In this example the product of the excesses is 14. Cast the 9's out of this product, and if the multiplication is correct, this excess will equal the excess found by casting out the 9's in the product of the example proper.

Check the product 593 × 3467 by casting out the 9's.

Further suggestions. 1. For speed drills, use the devices given in Suggestion (1) for addition on p. 48. One of these drills may be extended so as to prepare for the carrying idea in multiplication examples.

× 5, + 2	
5	27
7	
6	
etc.	

Suppose 5 is the multiplier and 2 is to be added. The pupil multiplies 5 by 5 and adds 2 to the product, obtaining 27. A drill that more nearly meets the conditions in written work where carrying is involved is obtained by writing the multiplicands in a horizontal row as follows:

5, 7, 6, 4, 8, 1, 3, 9, 2 | × 5

As the teacher points to any number of the row the pupils multiply it by 5 (in this case) and add the number previously assigned by the teacher. For instance, the teacher states that 2 is to be the number added. She then places the pointer, say, on 9 and the class gives the answer, obtained by multiplying 9 by 5 and adding 2 to the product.

2. In teaching the tables, make use of the law of commutation for multiplication. Thus $5 \times 7 = 7 \times 5$.

3. Make use of the form

$$\begin{array}{r} 185 \\ \underline{20} \\ 3700 \end{array} \qquad \begin{array}{r} 185 \\ \underline{300} \\ 55500 \end{array}$$

4. In later work let the pupils, at least the brighter ones, memorize the squares of 13, 14, 15, 16, 25, and the multiplication tables of all numbers up to 20 where the products are at least as great as 100.

5. As a preparation for division tables, ask questions like 6 times what number gives 18?

DIVISION

Short division. 1. *The beginning.* Short division is begun soon after the first written work in multiplication. When the pupil learns $3 \times 4 = 12$, he can answer $3 \times ? = 12$, and is really dividing 12 by 3. The written form is $\begin{array}{l} 3\overline{)12} \\ \quad 4 \end{array}$ and the pupil says 3 is contained in 12 four times, 12 divided by 3 is 4, or four 3's are 12.[1] Longer examples, in which there are no remainders in the steps, follow next. It is not necessary to give children at this age any reasons, but some teachers may prefer to give the following development: 693 is made up of $600 + 90 + 3$. In dividing this by 3, we have $\begin{array}{l} 3\underline{)600 + 90 + 3} \\ \quad 200 + 30 + 1 \end{array}$. From this the abbreviated form is derived.

2. *Remainders in the steps.* When there are remainders in the steps, there should be a preliminary oral drill like: How many 3's are there in 25 and what is the re-

[1] Some teachers prefer the overhead quotient, as in long division.

mainder? It will be found helpful to write on the board some numbers not exactly divisible by a chosen divisor. Write 34, 29, 65, 44, 52, 76, and others. Choose 7 for a divisor. The teacher places the pointer at 44 and the pupil answers "6 and 2 over." It is well to choose divisors that will be used in the written work.

The class is now ready for examples like these: Divide 357 by 3; 2716 by 4, etc. When there is a final remainder, write it in the usual fractional form in the quotient.

Do not allow the writing in of figures in the dividend during the process of dividing, for the habit leads to slow work and spoils the appearance of the example.

Long division. 1. *The beginning.* The safest plan for the inexperienced teacher in beginning long division is to choose a divisor that necessitates the longer process. There is some advantage in choosing for a first example one that could be done by short division, provided the two processes are compared side by side. There must, however, be no tedious delay in the matter.

The following suggestions may be helpful in connection with this plan.

Begin with an example in short division. Ex. Divide 37,232 by 8.

$$\begin{array}{l} 8\underline{)37232} \\ \quad 4 \end{array} \qquad\qquad \begin{array}{l} \quad\;\; 4 \\ 8\overline{)37232} \\ \quad \underline{32} \\ \quad\;\; 52 \end{array}$$

In long division we say 8 is contained in 37 four times, as in short division, but write the 4 above the 7. Instead of subtracting 32(4 × 8) mentally from 37, as in short division, we write the product, 32, under the 37 and subtract, writing the remainder, 5, as in ordinary subtraction. Then, instead of thinking the remainder 5 to the left of the 2 in the

dividend as in short division, we bring the 2 of the dividend down to the right of the 5. We then find how many times 8 is contained in 52, and proceed as before.

2. *Further preliminary suggestions.* Choose examples at first so that there will be remainders in the successive subtractions. Avoid zeros in the quotient at first. Choose divisors at first like 21, 42, 62, 93, rather than 12, 24, 26, 39. Also 201 before 241 and 831 before 381 or 318. This makes it less difficult to determine the figure in the quotient on account of the smaller figures being in the lower orders in the divisor.

Place the quotient above the dividend, writing the first figure of the quotient above the right-hand figure of the number first used in the dividend.

$$\begin{array}{r} 1 \\ 973\overline{)17624} \\ \underline{973} \\ 789 \end{array}$$

3. *Steps to be emphasized.* In teaching long division the teacher should be sure that the class can answer the following questions:

(*a*) How many figures in the dividend do we first divide by the divisor?
(*b*) What is the trial quotient resulting from this division?
(*c*) Where is the trial quotient figure placed?
(*d*) What two numbers are now multiplied?
(*e*) Where is the product placed?
(*f*) What is next done? (Subtract.)
(*g*) Is the remainder less than the divisor? If not, the trial quotient is too small.
(*h*) What is next done? (Bring down the next figure of the dividend).
(*i*) What is next done? (Divide, etc.)

Checks. A common way of checking in division is to multiply the quotient by the divisor and add the remainder to the product. The result should equal the dividend. To check by casting out the 9's, multiply the excess in the quotient by the excess in the divisor and add to the product the excess in the remainder. The excess in this sum should equal the excess in the dividend. The process is illustrated in the following example:

	556	7			
6	384)213563	2			
	1920		$7 \times 6 = 42$	6	
	2156			5	
	1920			11	2, which equals the excess in the dividend.
	2363				
	2304				
	59	5			

QUESTIONS

1. Why should short division precede long division?
2. Divide 142,674 by 376 and check by casting out the 9's.
3. Divide $16,632 by $24 and then by 24, retaining in each case the dollar sign wherever it logically belongs. Note that in the first case the quotient is abstract and hence its figures are taken as multipliers in obtaining the partial products. In the second case the divisor is abstract. Hence its figures are taken as multipliers in obtaining the partial products. See p. 30.

FACTORING

Fundamental ideas. The early drills in multiplication and division form the basis for factoring. Pupils first learn products like 2×6 and 3×4. Then they learn to divide 12 by 2, 3, 4, and 6. They are asked later to name all the divisors of 12. In so doing they are finding all the factors of 12. The pupil is to understand that if he is to find a

factor of a number, he must seek a divisor of that number. One factor of a number being found, another factor is the quotient obtained by dividing the given number by the factor first found. The product of these two factors gives the original number.

Let it be required to factor 112. A divisor of 112 is 4. Therefore one factor is 4. $112 \div 4 = 28$. Therefore $112 = 4 \times 28$. If the prime factors of 112 are desired, we have $112 = 2 \times 2 \times 2 \times 2 \times 7$.

The pupil must understand that to factor a number he writes the number equal to the product of those factors that produce the number. Note that the product of all the factors of a number does not equal that number. Thus $12 = 4 \times 3 = 6 \times 2 = 2 \times 2 \times 3$. While 2, 4, 3, and 6 are factors of 12, their continued product does not equal 12.

Tests of divisibility. Certain tests of divisibility aid in factoring. The most useful are:

1. All even numbers are divisible by 2.
2. A number is divisible by 4 if the number represented by the two right-hand digits is divisible by 4. To illustrate, 1932 is divisible by 4 because 32 is divisible by 4. Why is this true? (Suggestion: $1932 = 1900 + 32$.)
3. A number ending with a 5 is divisible by 5.
4. A number ending with a 0 is divisible by both 5 and 10.
5. A number is divisible by 9 if the sum of its digits is divisible by 9. For instance, 3465 is divisible by 9 since the sum of its digits, 18, is divisible by 9. The proof of this principle is found in the higher arithmetics.

6. A number is divisible by 3 if the sum of its digits is divisible by 3.

Questions

The following are exercises for the teacher:

1. When is a number divisible by 8? (Test No. 2 above should give a suggestion.)

2. When is a number divisible by 6, 15, 18, 24, 72? (Make use of the principle that the product of two or more factors of a number is also a factor of that number, provided those factors are prime to each other. Thus since 11 and 7 are factors of 1001, 77 is also a factor, and hence 1001 is divisible by 77. Note that 72 is divisible by 4 and 8, but not by 32 since 4 and 8 are not prime to each other.)

3. Write all the possible divisors of 27,720. Twenty-three divisors, besides the number itself and 1, should be obtained by using the standards laid down under this topic. If the reader knows the tests for 7's and 11's as divisors, he can obtain ninety-four different divisors in all.

GREATEST COMMON DIVISOR

Its value. There is very little occasion for finding the greatest common divisor in actual practice. It is of service in reducing fractions to their lowest terms, but even here one generally reduces fractions by successive divisions. Hence the long Euclidean method has no place in an elementary arithmetic.

Ideas involved. Pupils should work examples, bringing out in succession the ideas of divisor, common divisor, and greatest common divisor. The following questions may be asked:

(*a*) Name the divisors of 12; of 18.
(*b*) Name a divisor common to both 12 and 18. Name another.
(*c*) What is the greatest divisor common to both 12 and 18?

Written forms. The G. C. D. can frequently be found by inspection, but when this is impossible, either of the following forms may be used:

(*a*)
$$\begin{array}{r} 54 = 2 \times 3 \times 3 \times 3 \\ 72 = 2 \times 2 \times 2 \times 3 \times 3 \\ 108 = 2 \times 2 \times 3 \times 3 \times 3 \\ \hline \text{G. C. D.} = 2 \times 3 \times 3 = 18. \end{array}$$

(*b*)
$$\begin{array}{r|l} 9 & 54 - 72 - 108 \\ \hline 2 & 6 - 8 - 12 \\ \hline & 3 - 4 - 6 \\ \hline \multicolumn{2}{c}{\text{G. C. D.} = 9 \times 2 = 18.} \end{array}$$

In using the first form it will be found safest to factor each number into its prime factors. The pupils may, later, factor as follows:

$54 = 6 \times 9$	9 is seen to be a common factor and 2 is the
$72 = 8 \times 9$	G. C. D. of the other factors, 6, 8, and 12. The
$108 = 12 \times 9$	G. C. D. is therefore 9 times 2, or 18.

In using the second form divide by the greatest divisors that are seen to be common.

LEAST COMMON MULTIPLE

Its value. Considerable use is made of least common multiple in the addition and subtraction of fractions, but the work is generally simple. Hence avoid long tedious examples in L. C. M. It should be remembered, however, in both G. C. D. and L. C. M. that the pupils should understand well the fundamental ideas involved. Working merely by rule usually results in confusing these two topics.

Ideas involved. The pupils should first work examples in finding multiples of various numbers, then common multiples of two or more numbers, and lastly the least common multiple. Ask questions as follows:

(*a*) Give some multiples of 6. (*Ans.* 6, 12, 18, 24, 30, 36, 42, 48.)
(*b*) Give some multiples of 8. (*Ans.* 8, 16, 24, 32, 40, 48.)
(*c*) Give a common multiple of 6 and 8. Give another.
(*d*) What is the least common multiple of 6 and 8?

Frequently much useless work in finding the L. C. M. can be saved by discovering if some of the numbers given are factors of any of the others. Suppose it is required to find the L. C. M. of 2, 7, 3, 14, 15, 30. As 30 is divisible by 15, the 15 need not be any longer considered, for any number that is divisible by 30 is also divisible by 15. In like manner the 2 and the 3 should be eliminated, for they are also divisors of 30. Similarly 7 is a divisor of 14 and hence 7 need not be considered. The problem resolves itself into finding the L. C. M. of 14 and 30. Subject every example in L. C. M. to this test before using any set form in the written work.

Written forms. Either of the forms given below may be used.

Ex. Find the L. C. M. of 6, 8, 54, 72, 9, 108, 3. The 6, 8, 9, and 3, are all contained in the 54, 72, 108. Hence the last numbers mentioned are the only ones that concern us.

(*a*)

$$\begin{array}{rl} 54 &= 2 \times 3 \times 3 \times 3 \\ 72 &= 2 \times 2 \times 2 \times 3 \times 3 \\ 108 &= 2 \times 2 \times 3 \times 3 \times 3 \\ \hline \text{L. C. M.} &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 216. \end{array}$$

(*b*)

$$\begin{array}{r|l} 9 & 54 - 72 - 108 \\ \hline 2 & 6 - 8 - 12 \\ \hline 2 & 3 - 4 - 6 \\ \hline 3 & 3 - 2 - 3 \\ \hline & 1 - 2 - 1 \end{array}$$

$$\text{L. C. M.} = 9 \times 2 \times 2 \times 3 \times 2 = 216.$$

The first form is preferable to the second, for by using it the pupil gains a clearer understanding of L. C. M. Furthermore the second form is frequently confused with the G. C. D. It may prove a good plan to use form (*b*) as given under the topic Greatest Common Divisor and form (*a*) as above in L. C. M. In using the second form, it is advisable to use as large a divisor as possible, provided it

is contained in all the numbers. If not, it is safer to choose smaller divisors.

In using the first form, the pupil learns that in writing down the L. C. M. he must take each factor the greatest number of times it is found in any one of the numbers. Another good way, in using form (*a*) after all the numbers have been factored, is to write down first the factors of any one of the numbers, preferably the largest. The desired L. C. M. must contain these factors. Thus, in the above example write down again, beneath, the factors of 108. We have $2 \times 2 \times 3 \times 3 \times 3$. Looking at the factors of 72, we see that there is need of one more 2 in the answer, but the two 3's have been provided for. Looking at the 54, we see that the one 2 and the three 3's are all provided for. Hence the L. C. M. is $2 \times 2 \times 3 \times 3 \times 3 \times 2$, or 216.

COMMON FRACTIONS

Early work. The study of comparative magnitudes in the first grade is the basis for later work in fractions. In a strictly logical sense, the pupil should not add fractions until he has studied the least common multiple; but in practice he first works many simple examples in the addition and subtraction of fractions and also reduces simple fractions to higher or lower terms before learning to multiply or divide the numerator and denominator of a fraction by the same number.

Value of objects. The use of objects or diagrams is particularly advisable in connection with the study of fractions, but they should be laid aside when the purpose for which they are used has been realized. Some common

devices are the wooden apple, blocks, and colored circular disks which the teacher may make and attach to a large card or sheet of paper. The rectangle and circle are commonly used for board diagrams.

Fundamental ideas. One reason why pupils so often find the study of fractions difficult is that they do not fully understand what a fraction is. Fractions cannot be understood merely by studying the symbolism. The fundamental ideas are obtained from an objective representation of fractions.

Draw a line any convenient length. Divide it into four equal parts. Consider three of these parts. In formulating the thought that we are considering the three of the four equal parts, we are defining what is called

the fraction three fourths. The whole line is called the unit. Four, in this case, is called the denominator of the fraction, since it names the number of parts into which the unit is divided. Three is called the numerator, since it designates the number of equal parts that are here taken. Thus far we have not used in this explanation the symbols for a fraction, in order to make it clear that the fundamental ideas are obtained from things, not from symbols. Just as 5 is not the number five, but is the symbol for that idea, so $\frac{3}{4}$ is not the actual fraction, but represents a fractional part of a thing.

Decimal fractions. If the above unit is divided into a number of equal parts equal to some power of 10 and a number of these parts are taken for consideration, we have what are called decimal fractions. These may be

written in the form of common fractions or in the familiar decimal form.

REDUCTION OF FRACTIONS

Reduction to higher terms. Pupils first learn equalities like $\frac{1}{2} = \frac{2}{4}$, $\frac{1}{2} = \frac{3}{6}$, and $\frac{1}{3} = \frac{2}{6}$ by means of objects or diagrams. When it becomes necessary to deduce a rule for the reduction of fractions, obtain this by examining known equalities like those written above, in which one observes that the value of a fraction is not changed by multiplying or dividing the numerator and denominator by the same number. How is this principle applied?

Ex. Reduce $\frac{5}{7}$ to 21sts. This means that we wish to write $\frac{5}{7}$ equal to a fraction having 21 for a denominator. We first write $\frac{5}{7} = \frac{}{21}$. We next multiply the denominator 7 by 3 to get 21. We must then multiply the numerator 5 by the same number, 3, so as not to change the value of the fraction. Since $3 \times 5 = 15$, we write 15 for the numerator of the second fraction. Therefore $\frac{5}{7} = \frac{15}{21}$.

This is called reducing to higher terms, a somewhat contradictory expression, which is generally accepted. It may be necessary in examples like the above to tell the pupils to divide 21 by 7 to get the number by which we multiply the numerator 5, but it should be emphasized that both the numerator and denominator of $\frac{5}{7}$ are multiplied by 3.

Reduction to lower terms. In reducing fractions to lower terms, pupils are generally required to reduce them to their lowest terms. This is usually done by successive reductions, as in $\frac{54}{72} = \frac{6}{8} = \frac{3}{4}$. If any divisor is written in, it should not be written at the left of the fraction, but should be placed in both numerator and denominator. In practi-

cal work, it is commonly necessary to reduce fractions to specified lower terms.

Ex. Reduce $\frac{15}{48}$ to 16ths. First write $\frac{15}{48} = \frac{}{16}$. We divide 48 by 3 to get 16. Hence we must divide 15 by 3 to get the numerator of the second fraction.

A working rule. In practical work, such as might arise in drawing to scale in manual training, complex fractions may arise. Let it be required to draw a line to represent 9 ft. 10 in., the scale being 1 inch to represent a foot. A ruler graduated to 16ths is to be used. It follows that the line must be $9\frac{5}{6}$ in. long. It becomes necessary to reduce $\frac{5}{6}$ to 16ths. We get $\frac{5}{6} = \frac{13\frac{1}{3}}{16}$. In drawing the line, a third of one 16th is approximated beyond 13 of of the 16 parts into which the inch is divided. Work like this adds much to a full understanding of fractions.

A good rule for reducing fractions both to higher and to lower terms may be stated as follows: Multiply the denominator of the required fraction by the numerator of the given fraction and divide by the denominator of the given fraction (using cancellation form). The work on the the right-hand side below is to be done before the numerator of the required fraction is written in.

$$\frac{5}{6} = \frac{13\frac{1}{3}}{16} \qquad \frac{5 \times \overset{8}{\cancel{16}}}{\underset{3}{\cancel{6}}} = \frac{40}{3} = 13\tfrac{1}{3}.$$

Suggestions relating to reduction and to fractions in general. 1. Have the class draw lines to scale as above.

2. Emphasize the fact that the denominator of a fraction names the number of equal parts into which the unit is divided and that the numerator tells the number of these

that are to be taken. Drawing work like that just mentioned should make this understood.

3. Make it clear that
   (*a*) Multiplying the numerator multiplies the fraction.
   (*b*) Multiplying the denominator divides the fraction.
   (*c*) Dividing the numerator divides the fraction.
   (*d*) Dividing the denominator multiplies the fraction.

4. One aspect of a fraction was shown in Suggestion 2. The pupil should understand also that a fraction is an indicated division. In the case of improper fractions the division can actually be performed. In the case of proper fractions the division idea has no significance until decimals are studied.

5. Remember that in the application of arithmetic simple fractions occur more frequently than hard ones.

6. Emphasize the use of cancellation. The pupils should understand when they cross out numbers that they are dividing certain factors in the numerator and denominator by the same number. When all the factors are canceled, why is the answer 1 and not 0?

ADDITION OF FRACTIONS

The first step. The fundamental principle in adding fractions is that the denominators must be equal. The first work, which is begun in the first two years, is oral and relates to objects. It is as easy to add three sevenths and two sevenths as to add three hats and two hats. Let the written form first be 3 sevenths + 2 sevenths = 5 sevenths; then use the common form, $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$.

Written forms. In adding fractions that are not mixed

numbers, it is common practice to write them horizontally in the equation form. It is customary to add mixed numbers in a column. Either of the accompanying forms will do for the latter case.

$$\begin{array}{l} 2\frac{2}{3} = 2\frac{4}{6} \\ 3\frac{1}{2} = 3\frac{3}{6} \\ 4\frac{1}{6} = \underline{4\frac{1}{6}} \\ \qquad\;\; 9\frac{8}{6} = 10\frac{1}{3}\ \textit{Ans.} \end{array} \qquad \text{or} \qquad \begin{array}{r|l} 2\frac{2}{3} & 4 \\ 3\frac{1}{2} & 3 \\ \underline{4\frac{1}{6}} & \underline{1\qquad\quad} \\ 9 & \frac{8}{6} = 1\frac{1}{3} \\ \underline{1\frac{1}{3}} & \\ 10\frac{1}{3}\ \textit{Ans.} & \end{array}$$

SUBTRACTION OF FRACTIONS

Written forms for the hard case. For the sake of uniformity, the form of the written work in the subtraction of fractions should be like that chosen in addition. The first of the accompanying forms gives nearly all the steps in detail and hence makes the work clear, but the second form is especially recommended on account of its brevity.

$$\begin{array}{l} 16\frac{1}{3} = 16\frac{2}{6} = 15\frac{8}{6} \\ \;\;3\frac{1}{2} = \;\;3\frac{3}{6} = \;\;\underline{3\frac{3}{6}} \\ \qquad\qquad\qquad\; 12\frac{5}{6}\ \textit{Ans.} \end{array} \qquad \text{or} \qquad \begin{array}{r|l} 16\frac{1}{3} = 16\frac{2}{6} & 8 \\ 3\frac{1}{2} = \underline{\;\;3\frac{3}{6}} & 3 \\ 12 & \frac{5}{6};\ 12\frac{5}{6}\ \textit{Ans.} \end{array}$$

The second form permits full use of the Austrian method of subtraction. The pupil can soon learn to place the 8 in the minuend very quickly. He first learns that in examples of the above type he must add 1 to the fractional part of the minuend, the 1 in this case being considered as $\frac{6}{6}$. In adding the $\frac{6}{6}$ to the $\frac{2}{6}$ he sees that he must add 6 and 2 to get the new numerator. A glance at the fraction $\frac{2}{6}$ in the second form above shows that the numerator and denominator of this fraction can be added to get the numerator that is written at the right of the vertical line.

MULTIPLICATION OF FRACTIONS

The multiplier a whole number. It is sometimes a difficult matter for the teacher to determine when it is proper to explain to young children the reasons for the different processes. It is a question of gradual development. But it seems proper that by the time the pupil is studying fractions in a serious way he should, as far as possible, understand the principles underlying the work. These may often be easily explained, as is shown in the following case in the multiplication of fractions. Let it be required to teach the class to multiply $3 \times \frac{6}{7}$. They know that 3×6 dollars equals 18 dollars. In like manner 3×6 sevenths equals 18 sevenths. Hence the rule is easily deduced.

Use cancellation as early as possible in the multiplication of fractions, as in $7 \times \frac{3}{14}$ or $\frac{7 \times 3}{14}$.

The multiplicand a whole number. In an example like $\frac{1}{8} \times 12$, where the multiplicand is a whole number, the times sign, to be exact, should be replaced by the word "of." The example is thus seen to be one in partition. In an example like $\frac{2}{8} \times 12$, first find $\frac{1}{8}$ of 12, using cancellation, and then multiply by 2, for $\frac{2}{8}$ of 12 is twice the value of $\frac{1}{8}$ of 12.

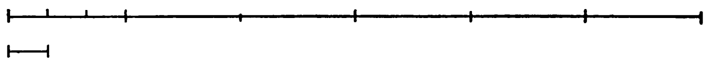
Relying upon his faith in the principles already established, the pupil will readily multiply, similarly, in examples where the multiplicand and multiplier are both fractions. Cancellation should be allowed in the form $\frac{4}{7} \times \frac{3}{8}$ as well as in the form of the single fraction, $\frac{4 \times 3}{7 \times 8}$.

DIVISION OF FRACTIONS

Fundamental ideas. In an example like $\frac{6}{8} \div \frac{2}{8}$ we have a case of division by measuring, as is readily seen if it is written in the form, 6 eighths ÷ 2 eighths. 2 eighths are contained in 6 eighths 3 times, just as a 2-foot rule is contained 3 times in 6 feet. The quotient in each case is the abstract 3. This idea should be made clear, for pupils are prone to write $\frac{6}{8} \div \frac{2}{8} = \frac{3}{8}$, which is incorrect.

The divisor a whole number. In an example like $\frac{2}{3} \div 2$, the measuring idea can be brought out, but it is better in practice to recall the fact that we can either divide by 2 or get $\frac{1}{2}$. Hence $\frac{2}{3} \div 2$ may be written $\frac{1}{2}$ of $\frac{2}{3}$ or $\frac{2}{3} \times \frac{1}{2}$, and the example becomes one in the multiplication of fractions.

The divisor a fraction. Where the divisor is a fraction, first consider an example like $6 \div \frac{1}{3}$. Make this objectively



clear. How many times may a rule $\frac{1}{3}$ of a foot long be marked off on one a foot long? (3 times.) Then how many times may a rule $\frac{1}{3}$ of a foot long be marked off on a line 6 ft. long? (6×3 times, or 18 times.) Hence $6 \div \frac{1}{3} = 6 \times 3 = 18$. This leads to the generalization that to divide by $\frac{1}{3}$ we multiply by 3.

Rule for inverting the divisor. Pupils generally work examples like the above before the generalized rule for inverting is taught. This rule may be developed as follows:

Continue the method illustrated in dividing 6 by $\frac{1}{3}$. 6 divided by 1 third is 18. 6 divided by 2 thirds is $\frac{1}{2}$ of 18 because the divisor has been doubled. Hence to divide 6

by $\frac{2}{3}$ we multiply 6 by 3 and divide by 2, which gives the same result as multiplying 6 by $\frac{3}{2}$. Observe that $\frac{3}{2}$ is $\frac{2}{3}$ inverted. The method is the same if the dividend is a fraction. Hence the rule, in order to divide by a fraction, multiply the dividend by the divisor inverted.

A second method of teaching the principle of inverting follows preliminary exercises in dividing fractions after reducing them to their least common denominator.

Ex. Divide $\frac{2}{3}$ by $\frac{3}{4}$.

SOLUTION: Reducing to the least common denominator and dividing, we have:

$$\frac{2}{3} \div \frac{3}{4} = \frac{8}{12} \div \frac{9}{12} = \frac{8}{9} \text{ (already understood).}$$

Then we work this example in multiplication —

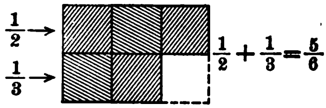
$$\frac{2}{3} \times \frac{4}{3} = \frac{8}{9}.$$

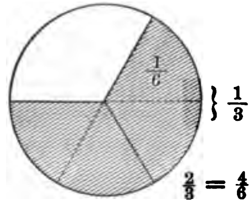
Since we get the same answer as when we divided $\frac{2}{3}$ by $\frac{3}{4}$ above, we conclude that in order to divide $\frac{2}{3}$ by $\frac{3}{4}$ we multiply $\frac{2}{3}$ by $\frac{4}{3}$.

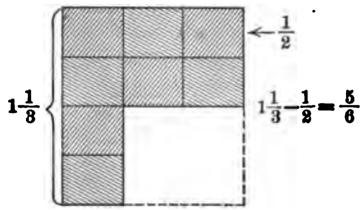
The pupil should be drilled in stating the two steps necessary in dividing, whether developed according to either the first or the second method given above. The first step is to write the dividend multiplied by the divisor inverted. Thus $\frac{2}{7} \div \frac{5}{6} = \frac{2}{7} \times \frac{6}{5}$. The next step is to multiply as in the multiplication of fractions. Thus $\frac{2}{7} \div \frac{5}{6} = \frac{2}{7} \times \frac{6}{5} = \frac{12}{35}$. In teaching the principle of inverting the divisor, the teacher will naturally choose that one of the two methods of developing employed above which is the better adapted to the text used.

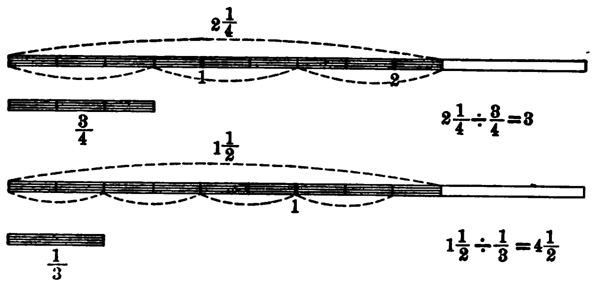
DIAGRAMS FOR FRACTION WORK

The following additional use of diagrams may prove suggestive in clearing up ideas regarding the processes. The answers may be read from the diagrams, without any figuring whatever.





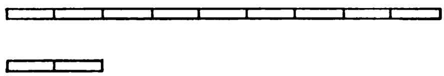




In the last diagram, notice that after measuring the divisor $\frac{1}{3}$ on the dividend 4 times, the remainder ($\frac{1}{6}$ of the unit chosen) is $\frac{1}{2}$ of the divisor. Hence the quotient is $4\frac{1}{2}$.

We may use the last diagram to illustrate that if the quantity to be measured (the dividend) and the quantity by which we measure (the divisor) are always of the same ratio, the quotient is always the same. By choosing different fundamental units and allowing in the several cases the same number of derived units in the divisor and the same

number in the dividend, we get different numerical values for the divisor and for the dividend, but always the same numerical value for the quotient. While this may at first seem complex, it is an illustration of the fundamental idea in division that we can divide by comparing the divisor and dividend only when they are expressed in terms of a common unit. In the diagram the common unit is the smallest unit represented.



In each case the dividend is made up of 9 of the smallest units and the divisor of 2 of them. First take the fundamental unit equal to one of these units. We then have the example:

$$9 \div 2 = 4\tfrac{1}{2}.$$

Choose the fundamental unit equal in length to the divisor. We have the example:

$$4\tfrac{1}{2} \div 1 = 4\tfrac{1}{2}.$$

Choose the fundamental unit equal to 3 of the smallest units. We have:

$$3 \div \tfrac{2}{3} = 4\tfrac{1}{2}.$$

Choose the fundamental unit equal to 4 of the smallest units. We have:

$$2\tfrac{1}{4} \div \tfrac{1}{2} = 4\tfrac{1}{2}.$$

Choose the fundamental unit equal to 5 of the smallest units. We have:

$$1\tfrac{4}{5} \div \tfrac{2}{5} = 4\tfrac{1}{2}.$$

Choose the fundamental unit equal to 6 of the smallest units. We have:

$$1\tfrac{1}{2} \div \tfrac{1}{3} = 4\tfrac{1}{2},$$

which is one of the examples illustrated in the preceding group of diagrams. The number of examples could be extended without limit.

Questions

1. Add $\frac{2}{3}$ and $\frac{1}{2}$ by means of a diagram, using a rectangle as the unit.

2. Find $\frac{2}{3}$ of $\frac{3}{4}$, using a circle as the unit.

3. Divide 2 by $\frac{2}{3}$, using a rectangle as the unit. Also divide $1\frac{2}{3}$ by $\frac{1}{2}$.

4. Using a scale of $\frac{1}{4}$, draw a line to represent a distance of 1 ft. 8 in.

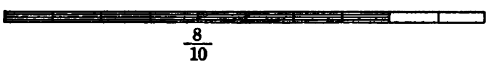
5. Write $2\frac{5}{32}$ in. in a form so that this distance can be laid off by means of a ruler graduated to 16ths. Do the same for $3\frac{5}{7}$ in.

6. Using a scale of 1 in. to 4 ft., draw a line to represent a distance of 13 ft. 4 in. Employ a ruler that is graduated to 16ths.

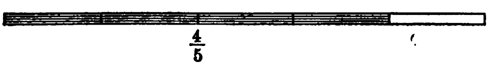
DECIMAL FRACTIONS

When begun. It seems best to follow the historic development and teach decimal fractions after common fractions, notwithstanding that there are good arguments for the reverse order. There should, however, be some understanding of decimals before the study of common fractions is completed.

Fundamental ideas. As commonly used, the terms "common fraction" and "decimal fraction" refer to the *form* of the written fraction. With respect to the *nature* of fractions, they may be classified as decimal and non-decimal; decimal where the unit (a line, for example) is divided into a number of equal parts represented by some power of ten, non-decimal where the unit is divided into a number of equal parts not some power of 10. How shall fractions be expressed in written form? All fractions may be written in the "common fraction" form, while decimal fractions may also be written after the manner of writing whole numbers. This latter form is meant when one speaks of writing decimals. The illustration shows two equal fractions, one decimal and the other non-decimal.



A decimal fraction, written $\frac{8}{10}$ or .8.



A non-decimal fraction, written $\frac{4}{5}$.

NOTATION AND NUMERATION OF DECIMALS

Methods of procedure. The notation and numeration of decimals is easily understood when related to the notation and numeration of whole numbers, and hence the subject should be taught in this connection. The plan of beginning by saying that we now have another way of writing fractions with denominator 10, 100, etc.,—for example, $\frac{1}{10}$ = .1 and $\frac{7}{10}$ = .7, — is open to serious objection. With this abrupt presentation the pupil does not understand that the writing of decimals is but a continuation of the decimal system adopted in the writing of whole numbers. Furthermore he is liable to get some absurd idea of a substitution of the point for the line and the written denominator.

Perhaps the best way to begin is to recall the writing of dollars and cents. Read $48.23. (48 dollars and 23 cents.) How many ten-dollar bills are expressed? (4.) How many dollar bills? (8.) Which is tens' place? Units' place? How many ten-cent pieces are expressed? (2.) How many cent pieces? (3.) One ten-cent piece is what part of a dollar? What place is occupied by the 2? (Tenths' place.) One cent is what part of a dollar? What place is occupied by the 3? (Hundredths' place.) 3 cents are what part of a dollar? 23 cents? Now read $48.23 as dollars and a fractional part of a dollar. (48 dollars and 23 hundredths of a dollar.)

In like manner read 138.20, 91.05, 29.2, .45, .05, .3, .1, and other decimals where the dollar or some other concrete quantity is understood to be the fundamental unit. Then read decimals where the kind of unit is not designated. Ask again for the names of the different orders

that have been given above. Before sending the class to the board, remind them to write decimal fractions in the form just learned and not in the common fraction form.

Perhaps the above will suffice as a foundation for drill work in the reading and writing of decimals, but the teacher may well introduce here some measuring work to fix ideas, using a ruler graduated to tenths. Draw a line 9 inches long. Extend it 7 tenths of an inch. How long is the whole line? How is this value written? Write decimals on the board and have the class draw the corresponding lengths.

Change of unit. An excellent drill, after the class is well started, is to write a mixed decimal fraction on the board and have the pupils read it with respect to the fundamental unit assigned for the moment by the teacher. Thus they read:

48.23 dollars.
482.3 ten-cent pieces.
4823 cent pieces.
4.823 ten-dollar bills (or pieces).

After the first number is on the board, the teacher says that we now choose the ten-cent piece as the fundamental unit. The class finds out that there are 482 ten-cent pieces and 3 cents besides. The number is written as above. In like manner the cent and the ten-dollar may be taken as the unit. The class should discover that the point is always at the right of the fundamental unit (units' place).

Instead of money in a drill like the above, the teacher may use the familiar colored sticks. Perhaps one of the

best means of teaching the reading and writing of decimals and of giving an excellent drill in varying the unit is to make use of the surveyor's chain or tape, the pupils making the readings and recording the lengths. This can, if necessary, be done in the schoolroom by considering a number of chain or tape lengths to have been already laid off and measuring to some mark on the floor.

Significance of the point. — The teacher will naturally call attention to the object of using the point in writing decimals. The customary explanation is that it is used to separate the whole number from the fraction. In view of the fact that the point has by universal consent been placed to the right of the units' order this explanation is a most natural one. But from some points of view it is unfortunate that the point has been thus placed. If placed above or below units' order, in order to designate the fundamental unit, a certain symmetry would result, which would help the pupil in reading decimals. For then tens and tenths, hundreds and hundredths, thousands and thousandths, etc., would be balanced with respect to the point. By the present system the pupil is apt to be confused by the fact that the second place to the left of the point is tens' place, while the first place to the right is tenths' place. Care should be taken in using the pointer that it is placed on the order that is read. If units' order had been distinguished as in the following illustration, pupils would find it easier to understand the reading and writing of decimals.

$$38\overset{\bullet}{4}57 \text{ or } 38\underset{\bullet}{4}57 \text{ or } 38\mathbf{4}57$$

Things to consider in the development. The following points should be considered in teaching the notation and numeration of decimals:

1. Begin by relating the work to the notation and numeration of whole numbers as shown above.

2. Begin with decimals that involve tenths and hundredths, after the manner given above, and also bring out the relations between any two adjacent orders. (10 times and $\frac{1}{10}$ of.)

3. Drill on reading and writing decimals involving tenths and hundredths.

4. Introduce the idea of thousandths. 1 tenth of a tenth is 1 hundredth; and we know that hundredths' place is next to tenths' place on the right. 1 tenth of a hundredth is 1 thousandth, and thousandths' place is naturally located next to hundredths' place on the right.

5. Drill on reading and writing decimals involving thousandths and lower orders. Then pass to higher orders.

6. Say ten-thousandths, not tens of thousandths.
Say one hundredth, not one one-hundredth.

7. In reading decimals, say "and" at the decimal point only. This guides the pupil in placing the point when he is given numbers to write.

8. Emphasize the following principle in reading decimals: Read first the whole number. Then read the fraction on the right of the point just as if it were a whole number, only giving in addition the name of the order occupied by the right-hand digit. Thus in reading 3625.3625, say three thousand six hundred twenty-five and three thousand six hundred twenty-five ten-thousandths. The primary unit,

one, is not expressed, while the secondary unit, ten-thousandths (of one) is expressed. The necessity of using both of these units is seen in practical work. Read 28.37 ft. We say, in effect, 28 feet and 37 hundredths of a foot. Here the primary unit is a foot and the secondary unit is a hundredth of a foot. Have enough of this fundamental work relating to the units employed to make the meaning of decimals clear.

9. Make it clear that annexing zeros to the right of a decimal does not change its value.

10. In writing a column of decimals, insist that the points be placed in column and that the same be done with the corresponding orders. This puts the work in proper form for the addition and subtraction of decimals.

11. Give sufficient drill so that the pupils shall have the position of the different orders firmly fixed in mind. They should know, for example, that thousandths' place is the third place to the right of the point without having to enumerate from tenths' place.

Questions

1. Read $3125 expressed in terms of ten-dollar bills. *Ans.* 312.5 ten-dollar bills. In terms of hundred-dollar bills. In terms of ten-cent pieces. In terms of one-cent pieces. *Ans.* 312,500 cents.

2. Read 145.67 in terms of tens. *Ans.* 14.567 tens. In terms of hundreds. Tenths. Hundredths. *Ans.* 14,567 hundredths.

3. How many tenths of an inch are there in a line 35.6 inches long? *Ans.* 356 tenths of an inch. How many hundredths of an inch?

4. State an easy rule for changing the position of the point when the unit is changed, as in the above examples. Notice that the value of the number, in each case, is not changed.

5. Compare the values of 57, 570, and 057. Also of .57, .570, and .057. State the principles involved.

MULTIPLICATION OF DECIMALS

Multiplying and dividing by 10, 100, etc. Two of the earliest and most useful principles in decimals relate to multiplying and dividing by 10, by 100, and by higher powers of 10. It is not easy to explain the principles involved in the operations with decimals without making the work too difficult for the average pupil, but, wherever possible, light should be thrown on the ideas involved. It will perhaps be sufficient to explain the two principles mentioned as follows:

We know that $58 \times 10 = 580$, $58 \times 100 = 5800$, and $58 \times 1000 = 58{,}000$. Write these in the form $58.0 \times 10 = 580$, $58.00 \times 100 = 5800$, and $58.000 \times 1000 = 58{,}000$. We observe that the answers may be obtained by moving the point in the dividend one place to the right when multiplying by 10, two places when multiplying by 100, and three places when multiplying by 1000. Then illustrate the same rule for decimal multiplicands: $58.25 \times 10 = 582.5$; $4.578 \times 100 = 457.8$.

Since $56.45 \times 10 = 564.5$, therefore $564.5 \div 10 = 56.45$ because multiplication and division are inverse processes. It is evident that in dividing by 10 we may obtain the answer by moving the point in the dividend one place to the left. In like manner the rules for dividing by 100, by 1000, etc. may be deduced.

A more logical method is to reduce the multiplicand and the dividend, in the above cases, to common fractions, multiply and divide according to the rules of multiplication and division of common fractions, change the results to decimals, and deduce the rule. To illustrate, $78.25 \times 10 = 78\frac{25}{100} \times 10 = \frac{7825}{100} \times 10 = \frac{7825}{10} = 782.5$. Compare the last number with the multiplicand, 78.25, and deduce the rule.

The multiplicand a decimal, the multiplier a whole number. In a product like 4×13.45, where the multiplier is a whole number, point off as in $4 \times \$13.45$, with which the class is already familiar. The general rule may be stated that in multiplying a decimal by a whole number we point off from the left in the product as many places as there are in the multiplicand. Another way is to add 13.45 four times, observe the number of places in the sum, and deduce the rule. This method is logical, for the principle always follows, no matter how many decimal places there are in the multiplicand. It would be inconvenient to use a multiplier of two or more figures in the second method on account of the length of the column.

The principle can also be deduced logically by reducing the multiplicand to a common fraction, multiplying by the given multiplier, changing the product to a decimal, and deducing the rule by comparing the product with the multiplicand.

The multiplicand and the multiplier both decimals. The explanations that may be given for pointing off where the multiplicand is a whole number and the multiplier a decimal are similar to those where multiplicand and multiplier are both decimals. Hence we shall confine ourselves to the latter case. The teacher may find it expedient to give the pupils the rule at first and at a later time explain the principles involved. The following method may be used for developing the rule:

Ex. Multiply 18.37 by 3.775.

First multiply 18.37 by the whole number 3775, obtaining 69,346.75. Since 3.775 is $\frac{1}{1000}$ of 3775, the answer in 3.775×18.37 is $\frac{1}{1000}$ of

69,346.75, or 69.34675, which is found by moving the point in the first product three more places to the left. Now compare the number of decimal places in the product 69.34675 with the total number in 18.37 and 3.775. The number is the same. Show by other examples that this is always true. Hence the rule that the number of decimal places in the product equals the sum of the decimal places in the multiplicand and the multiplier.

18.37	18.37
3775	3.775
9185	
12859	
12859	
5511	
69346.75	69.34675

Two other methods follow:

(*a*) Ex. Multiply 18.37 by 3.775.

Expressing the numbers to be multiplied as common fractions, we have $18\frac{37}{100} \times 3\frac{775}{1000}$. This becomes $\frac{1837}{100} \times \frac{3775}{1000} = \frac{6934675}{100000}$, or in decimal form 69.34675. Deduce the rule as before.

(*b*) Ex. Multiply 18.37 by 3.775.

Multiply the numbers as given, the product without the point being 6934675. Looking at the integral parts of the numbers multiplied, we see that we have multiplied 18 and a fraction by 3 and a fraction. The product must be something over 3 × 18, or something over 54. Looking at the 6934675 we see that the answer must be something over 69 and not as great as 693. Hence the point must be placed to the right of the 9. Therefore the answer is 69.34675. This will serve in many cases as a general method of procedure, no other rule being necessary; but in a product like 45.674 × .000675 this method is not easily applied. In case the teacher uses the last method to establish the rule for pointing off, as soon as the point has been placed after the 9 she should have the class compare the number of decimal places to the right of the point in the product with the total number in the multiplicand and the multiplier and deduce the ordinary rule.

Questions

1. How many places are there to the right of the point in the product where 67.754 is multiplied by .57? 4.45 by 13.7? .098 by .4654? Do not perform the multiplications.

2. How many places are there to the left of the point where 3.56 is multiplied by 4.567, pointing off from the left as suggested in the first

part of (*b*) above? Where 2.6785 is multiplied by 27.1? 16.6785432987 by 2.456378456? Do not perform the multiplications.

3. Knowing that $26475 \times 3146 = 83290350$, point off in the following, using the method in (2) when practicable:

26.475×3.146	$.026475 \times 31.46$
2.6475×3.146	$264.75 \times .003146$

4. The value of a product is not changed if the points in the multiplicand and the multiplier are moved the same number of places in opposite directions. Thus, $18.25 \times 3.456 = .1825 \times 345.6 = 1825 \times .03456$. This principle often enables one to point off from the left, as in (2) above. For instance, in $345.67 \times .0375$, we write $345.67 \times .0375 = 3.4567 \times 3.75$. The product will have two places to the left of the point.

How many places to the left of the point are there in the following products:

$567.67 \times .0665$?	$56.98 \times .1167$?
$.0034 \times 46856.2$?	$.03467 \times 568.7643$?

DIVISION OF DECIMALS

The divisor a whole number. Three main points need to be considered in case the divisor is a whole number:

1. Place the first figure of the quotient as when dividing whole numbers; that is, place it under the right-hand figure of the number first used in the dividend if in short division and over it if in long division.

2. Place the point in the quotient directly below (or above) the point in the dividend.

3. It is sometimes necessary to annex zeros to the right of the point in the dividend when the dividend is a whole number, or to the right of the decimal part of the dividend.

It is a good plan to place the point in the quotient at the start. In any case do not delay putting it in until after all the quotient figures have been written. The second

example following shows the advantage in placing the point in the quotient at first, followed by the zero, before writing the first significant figure, 1.

```
                       .0125
8)62.00          264)3.3000
  7.75               2 64
                       660
                       528
                      1320
                      1320
```

The divisor a decimal. In case the divisor is a decimal, but one preliminary step is needed to reduce the example to the above type, this step being to multiply both divisor and dividend by the power of ten that will make the divisor a whole number. The work is done mechanically as follows:

1. Rid the divisor of decimals by moving the point to the right of the right-hand figure.

2. Move the point in the dividend as many places to the right as the point in the divisor was moved.

3. Place the point in the quotient above the new position of the point in the dividend.

Ex. Divide 16.824 by .24. By following the above suggestions, the example is reduced to dividing 1682.4 by 24. The pupil counts the number of places to the right of the point in the divisor. He then counts the same number of places to the right of the point in the dividend. He observes that the point should be placed, in the above example, between the 2 and the 4 of the dividend. He locates the point in the quotient above the space between the 2 and the 4. Then he continues the division.

```
       70.1
.24)16.82 4
    16 8
         2 4
         2 4
```

In case it is desired to retain the original decimal points in the divisor and dividend the following form may be used, where the caret is placed at the new position of the point.

$$\begin{array}{r} 70.1 \\ .24_{\wedge})\overline{16.82_{\wedge}4} \\ \underline{16\ 8} \\ 24 \\ \underline{24} \end{array}$$

QUESTIONS

1. Give, without dividing, the number of figures to the left of the point in the following examples: 688 divided by 2.15; 38.2 by 19.3; 8.175 by 2.35. SUGGESTION: Divide mentally the integral parts of the numbers.

2. How many figures are there to the left of the point in dividing 3.22 by .08? .12 by .003? .012 by .003? SUGGESTION: First rid the divisor of decimals.

CHANGES IN THE FORM OF FRACTIONS

Decimal to common form. We have already considered the reduction of common fractions to higher and to lower terms. Decimal fractions may be similarly reduced. Before considering the reduction of decimals we shall take up the changing of fractions from the decimal to the common form, and *vice versa.* The first of these changes is simple and has already been made use of under the topic "Multiplication of decimals." Two classes of examples will suffice here.

Ex. Change .875 to the common fraction form.

We look at .875, think 875 thousandths, and write $\frac{875}{1000}$. The answer is not yet in the proper form, for the rule should be that common fractions must always be reduced to their lowest terms unless there is some special reason for not so doing. The work then is $.875 = \frac{875}{1000} = \frac{7}{8}$.

Ex. Change $.33\frac{1}{3}$ to the common fraction form.

We write $.33\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3} \times \frac{1}{100} = \frac{100}{3} \times \frac{1}{100} = \frac{1}{3}$.

Common to decimal form. Changing a fraction from the common to the decimal form can be effected in either of the following ways:

(*a*) First change to an equivalent common fraction with a denominator some power of ten. Then write in decimal form.

Ex. Change $\frac{4}{5}$ to 10ths, written in the decimal form.

First write $\frac{4}{5} = \frac{8}{10}$. Think 8 tenths and write .8. The mechanical work is, $\frac{4}{5} = \frac{8}{10} = .8$.

Exs. $\frac{3}{125} = \frac{24}{1000} = .024$; $\frac{1}{50} = ?$ $\frac{1}{250} = ?$

This method is especially good in the case of simple fractions whose denominators are divisors of some power of 10. The following method is to be recommended for general practice:

(*b*) Consider the fraction as an indicated division and divide the numerator by the denominator.

Ex. Change $\frac{4}{5}$ to the decimal form.

$\frac{4}{5} = .8$. SIDE WORK: $5\overline{)4.0}$ .8

Also, $\frac{1}{500} = .002$. SIDE WORK: $500\overline{)1.000}$ .002

Reduction of decimals. It is frequently necessary to change decimals to equivalent decimals having higher or lower terms, as in percentage where it is necessary to express decimals as hundredths. A few examples will illustrate the method in each case:

(*a*) Reduce to higher terms: .6 to hundredths; $.07\frac{1}{2}$ to a pure decimal; $.05\frac{3}{4}$ to a pure decimal.

SOLUTIONS: $.6 = .60$.

$.07\frac{1}{2} = .075$, because $\frac{1}{2}$ of .01 is .005.

$.05\frac{3}{4} = .0575$, because $\frac{3}{4}$ of .01 is .0075.

In this connection it should be noticed that the common-fraction part of the decimal belongs to the order of the figure on the left, as is shown in the diagram.

.6			
.6	0		
.0	$7\frac{1}{2}$		
.0	7	5	
.0	$5\frac{3}{4}$		
.0	5	7	5

(*b*) Reduce to lower terms: .065 to hundredths; .005 to hundredths; $.333\frac{1}{3}$ to hundredths; .0175 to hundredths; .04625 to thousandths.

SOLUTIONS:

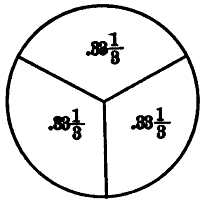
$.065 = .06\frac{1}{2}$, for the .005 is $\frac{5}{10}$ $(\frac{1}{2})$ of .01.
$.005 = .00\frac{1}{2}$, for a similar reason.
$.333\frac{1}{3} = .33\frac{1}{3}$, for the $.003\frac{1}{3}$ is $\frac{3\frac{1}{3}}{10}$ $(\frac{1}{3})$ of .01.
$.0175 = .01\frac{3}{4}$, for the .0075 is $\frac{75}{100}$ $(\frac{3}{4})$ of .01.
$.04625 = .046\frac{1}{4}$, for the .00025 is $\frac{25}{100}$ $(\frac{1}{4})$ of .001.

Decimal answers should not, as a rule, contain the common-fraction form. For example, write 5.35 instead of $5.3\frac{1}{2}$. An answer like $45.33\frac{1}{3}$, where the combination forms one of the familiar "aliquot parts" used in business, is usually accepted, but not one like $43.3\frac{1}{3}$ or $43.333\frac{1}{3}$.

Aliquot parts. Certain decimals reduce to simple common fractions that shorten the work in most examples. These aliquot parts should be thoroughly memorized. The most usable are:

$.12\frac{1}{2} = \frac{1}{8}$	$.25 = \frac{1}{4}$	$.40 = \frac{2}{5}$	$.62\frac{1}{2} = \frac{5}{8}$	$.80 = \frac{4}{5}$
$.16\frac{2}{3} = \frac{1}{6}$	$.33\frac{1}{3} = \frac{1}{3}$	$.50 = \frac{1}{2}$	$.66\frac{2}{3} = \frac{2}{3}$	$.83\frac{1}{3} = \frac{5}{6}$
$.20 = \frac{1}{5}$	$.37\frac{1}{2} = \frac{3}{8}$	$.60 = \frac{3}{5}$	$.75 = \frac{3}{4}$	$.87\frac{1}{2} = \frac{7}{8}$
	$.14\frac{2}{7} = \frac{1}{7}$		$.11\frac{1}{9} = \frac{1}{9}$	

The division of a circle into the desired number of equal parts, as in the diagram, will help to fix these ideas.



SOME SHORT METHODS

Value of abbreviated methods. Abbreviated methods should not be learned when the time and effort spent in mastering them are not commensurate with the use made of them. Certain short processes, however, have a value and should be employed.

Addition and subtraction. 1. See suggestions on regrouping, previously given under addition and subtraction.

2. In case the L. C. M. of the denominators of two fractions is their product, find the numerator in the sum of the fractions by multiplying the numerator of each by the denominator of the other and adding the two products. Find the denominator by getting the product of the two denominators.

Ex. $\frac{2}{3} + \frac{3}{4} = \frac{17}{12}$. The sum of 2×4 and 3×3 is 17. The product of 3 and 4 is 12.

3. Add two fractions with numerators 1 by adding the denominators to get the numerator of the sum and multiplying the denominators to get the denominator of the sum.

Ex. $\frac{1}{5} + \frac{1}{11} = \frac{16}{55}$.

4. In the subtraction of mixed numbers, where the fraction part of the subtrahend is larger than the fraction part of the minuend, subtract as in the following example.

Ex. $16\frac{2}{6}$
$\underline{8\frac{3}{6}}$
$7\frac{5}{6}$

To get the numerator in the answer, subtract the 3 in the numerator of the subtrahend from the sum of 6 and 2, adding these numbers mentally just as they stand in the minuend.

Multiplication. 1. Apply the principle of regrouping in multiplication.

Ex. Multiply 25 by 7 by 4. This is done mentally if thought of in the form, $25 \times 4 \times 7$.

Ex. Multiply 25 by 36.

2. To multiply by 25, multiply by 100 and divide by 4.

Ex. $25 \times 832 = 20{,}800$. SIDE WORK: $4\overline{)83.200}$ 20,800

Ex. $25 \times 6.1 = 152.5$. SIDE WORK: $4\overline{)610.0}$ 152.5

To multiply by 50, multiply by 100 and divide by 2.

To multiply by $33\frac{1}{3}$, multiply by 100 and divide by 3.

State rules for multiplying by $12\frac{1}{2}$, $16\frac{2}{3}$, and other so-called aliquot parts of 100.

3. To multiply by 125, multiply by 1000 and divide by 8.

4. To multiply by $1\frac{1}{3}$, add $\frac{1}{3}$ of the multiplicand to itself.

Ex. $1\frac{1}{3} \times 256 = 341\frac{1}{3}$. SIDE WORK: $3\overline{)256}$ $\underline{85\frac{1}{3}}$ $341\frac{1}{3}$

Make up similar rules for multiplying by $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{6}$.

5. To multiply by $133\frac{1}{3}$, multiply by 100 and add to the product $\frac{1}{3}$ of itself.

Ex. $133\frac{1}{3} \times 26.5 = 3533\frac{1}{3}$. SIDE WORK: $3\overline{)2650}$ $\underline{883\frac{1}{3}}$ $533\frac{1}{3}$

6. To multiply by fractions like $\frac{2}{3}$, $\frac{3}{4}$, $\frac{12}{13}$, where the numerator is one less than the denominator, proceed as in the following example.

Ex. $\frac{12}{13}$ of $276 = 254\frac{10}{13}$.

Subtract from the multiplicand $\frac{1}{13}$ of itself.

SIDE WORK:
$$\begin{array}{r} 13\underline{)276} \\ \underline{21\frac{3}{13}} \\ 254\frac{10}{13} \end{array}$$

7. To multiply by 9, subtract the multiplicand from the product obtained by multiplying it by 10. (Of value only where the multiplicand is a large number.)

Ex. $9 \times 26.2 = 235.8$.

SIDE WORK:
$$\begin{array}{r} 262.0 \\ \underline{26.2} \\ 235.8 \end{array}$$

8. To multiply by 99, subtract the multiplicand from the product obtained by multiplying it by 100.

State a rule for multiplying by 999.

9. To multiply by 11, add each digit of the multiplicand to the one next to it, beginning on the right, first writing the right-hand digit, and finally the left-hand one. "Carry" as in ordinary addition.

Ex. $11 \times 2645 = 29{,}095$.

EXPLANATION: First write down the 5, the right-hand digit of the multiplicand. Then add 5 and 4, and place the sum, 9, to the left of the 5. Add 4 and 6 and place the digit 0 to the left of the 9, add the 1 to the sum of 6 and 2, and write their sum, 9, to the left of the 0. Then write the left-hand digit, 2, of 2645 to the left of the 9.

10. It is sometimes convenient, as in mental work, to multiply in succession by the factors of the multiplier.

Ex. Multiply 28 by 32.

SOLUTION:
$$\begin{array}{rl} 112 & = 28 \times 4. \\ \underline{8} & \\ 896 & \textit{Ans.} \end{array}$$

The 112 is written down after looking at the 28 in the example and multiplying it by 4.

11. In mental work it is sometimes better to multiply the higher orders first.

Ex. Multiply 23 by 5.

SOLUTION: First multiply 20 by 5 and add 15.

12. Where 11, 12, 13, etc., form the digits of the multiplicand, make use of the corresponding multiplication tables.

Ex. Multiply 15,314 by 3.

SOLUTION: Say 3 times 14 is 42, writing down the 4 of the 42 first; 3 times 3 is 9; 3 times 15 is 45, writing down the 4 of the 45 first.

13. A serviceable short cut is illustrated in the following examples:

		319283
8263	8263	56832
287	728	2554264
57841	57841	10217056
231364 = 4 × 57841	231364	17879848
2371481	6015464	18145491456

Notice the position of the right-hand figures of the partial products. In the first two examples use is made of the fact that 28 in the multiplier equals 4 times 7. In the third example, that 32 is 4 times 8, and that 56 is 7 times 8.

14. Rule for squaring numbers, especially for mental work:

Add to and subtract from the number to be squared such a number as will give, either in the sum or the difference obtained, a number ending with a 0. Multiply the numbers thus obtained, and add to the product the square of the number added and subtracted.

Ex. Square 18.

SOLUTION:

$$\begin{array}{rl} 16 & = 18 - 2 \\ \underline{20} & = 18 + 2 \\ 320 & \\ \underline{4} & = \text{the square of } 2. \\ 324 & \textit{Ans.} \end{array}$$

Ex. Square 99.

SOLUTION:

$$\begin{array}{rl} 9800 & \\ \underline{1} & \\ 9801 & \textit{Ans.} \end{array}$$

The algebraic formula, $a^2 = (a - b)(a + b) + b^2$, furnishes the proof for the above principle.

Division. 1. To divide by 25, multiply by 4 and divide by 100.

To divide by 50, multiply by 2 and divide by 100.

To divide by $33\frac{1}{3}$, multiply by 3 and divide by 100.

State rules for dividing by $12\frac{1}{2}$, $16\frac{2}{3}$, and other aliquot parts of 100.

Ex. $185 \div 33\frac{1}{3} = 5.55$.

SIDE WORK:

$$\begin{array}{r} 185 \\ \underline{3} \\ 555 \end{array}$$

Point off two places, giving 5.55.

2. To divide by 125, multiply by 8 and divide by 1000.

3. It is sometimes better to divide in succession by the factors of the divisor.

Ex. Divide 315 by 21.

SOLUTION: Divide by the first factor mentally unless it is desired to retain all the work. Divide 315 by 7 and the result by 3. All the work that need be written is: 45; 15. *Ans.*

Ex. Divide 423 by 54. SOLUTION: 47; $7\frac{5}{6}$. *Ans.*

The method is not good where no factor of the divisor is a factor of the dividend.

4. Omit writing down the products in long division. As each digit of the divisor is in turn multiplied by the quotient digit, mentally subtract from the proper digits in

the dividend, writing down the remainder. The Austrian method of subtraction is well adapted for this kind of work.

$$\begin{array}{r} 4 \\ 573\overline{)25357} \\ 243 \end{array}$$

Thus in dividing 25,357 by 573 we first find 4, the quotient figure. Then say 4 times 3 is 12 and 3 are 15, writing down the 3 under the 5; 1 to carry; 4 times 7 is 28 and 1 (carried) are 29 and 4 are 33, writing down the 4; 3 to carry; 4 times 5 is 20 and 3 (carried) are 23 and 2 are 25, writing down the 2. Next bring down the 7 of the dividend to the right of the 243 and after obtaining the new quotient figure proceed as before.

5. In examples like the following, divide as indicated:

Ex. Divide 5.1 by 300. SOLUTION: $300\underline{)5.1}$
.01 7

Divide by 3 and move the point in the result two places to the left.

Ex. Divide 364 by 17,000.

SOLUTION:

$$\begin{array}{r} .021 \\ 17000\overline{)364} \\ 24 \\ 7 \end{array}$$

Divide by 17 and move the point in the result three places to the left.

In general move the point as many places to the left as there are zeros in the divisor, the divisor not being a decimal.

6. Ex. Divide $373\frac{2}{7}$ by 8.

SOLUTION: $8\underline{)373\frac{2}{7}}$
$46\frac{37}{56}$

8 is contained in 373 forty-six times with the remainder 5. Multiply the 5 by the denominator 7 in the fractional part of the dividend and add to the product the numerator 2. Divide this result by 8 times the denominator 7, and we have the fractional part of the answer. This is simply the result of reducing the complex fraction $\frac{5\frac{2}{7}}{8}$ to a more simple fraction, and is to be done mentally.

APPROXIMATIONS

If pupils were taught to approximate answers mentally either before or after solving, but especially before, teachers would encounter a far less number of ridiculous answers.

The illustrative examples that follow are typical of what can be done in percentage, interest, denominate numbers, and other topics.

EXAMPLES:

1. How much will 12 pairs of shoes cost at \$4.95 per pair?

SOLUTION: One pair costs nearly \$5, hence 12 pairs cost a little less than \$60.

2. If 7 hats cost \$24.50, what is the cost of 29 hats?

SOLUTION: Use 28 instead of 29 and 25 instead of 24.50. 28 hats cost 4 times the cost of 7 hats. The approximate answer is $4 \times \$25$, or \$100.

3. Multiply 3.56 by .21.

SOLUTION: .21 is a little more than $\frac{1}{5}$. The approximate answer is $\frac{1}{5}$ of 3.5, or .7.

4. Divide 2150 by 2.34.

SOLUTION: 2.34 is a little more than $2.33\frac{1}{3}$, or $2\frac{1}{3}$. By dividing 2100 by $\frac{7}{3}$ we get 900 as an approximate answer.

5. Multiply, approximately, 24 by $66\frac{1}{2}$; 32 by $.37\frac{1}{2}$; $.16\frac{2}{3}$ by $24\frac{3}{4}$.

6. Divide, approximately, 18 by .33; 1.5 by 1.49; 12.01 by 1.19.

7. Find the approximate value of $\frac{1600 \times .37\frac{1}{2} \times 259}{1.98 \times 13}$.

DEGREE OF ACCURACY IN ANSWERS

Judgment in the degree of accuracy desired in answers. Not only should pupils be trained to approximate answers in order to check absurd results, but they should also know the degree of accuracy to which the answers should be carried out. In a sense it is as ridiculous for a pupil to have \$45.852 as the final form of his answer, where \$45.85 is the correct commercial value, as to have an answer ten times too great. A fundamental principle in computation is that the answer cannot be any more accurate than the data warrant. For example, a surveyor who measures with his chain to the nearest link does not seek an answer

any closer than a whole number of links. A man would not think of measuring the sides of his rectangular field to the nearest inch. Why should he compute the length of the diagonal any more closely than he measures the sides? In "money" problems the case is somewhat different, for the data to be used are not subject to an individual error. The number of dollars and cents given is an exact measure of the real dollars and cents. The computation may give mills or tenths of mills in the answer and the answer may be absolutely correct, but business practice asks for nothing smaller than cents, and hence the final form of answers in such problems should show no order smaller than hundredths.

Writing numbers to a specified degree of accuracy. To express an answer correct to a certain number of decimal places, proceed as in the following examples:

Ex. Express 2.384 correct to hundredths (to .01).

The 4 being less than 5, the value of the decimal part of 2.384 is nearer .38 than .39. Hence we have 2.384 = 2.38 (correct to .01).

Ex. Express 45.678 correct to hundredths.

We have 45.678 = 45.68 (correct to .01), for the 8 in .678 being greater than 5, the value of .678 is nearer .68 than .67.

Ex. Express 34.6815 correct to thousandths.

Since the final figure on the right is 5, there is a doubt whether to take 34.681 or 34.682 as the approximate value. Computers have agreed to "give" instead of "take" where the figure is a 5. Hence we have 34.6815 = 34.682 (correct to .001).

Ex. Express 567.4569 correct to tenths. *Ans.* 567.5.

Ex. Express 67.6849 correct to hundredths. *Ans.* 67.68.

Working for a specified degree of accuracy in multiplication. In case numbers that have several decimal places are to be multiplied and it is known in advance how many

places are required in the product, the following method may be used:

1. Write the digits of the multiplier in reverse order, placing units' digit directly under the order of the multiplicand that is one place to the right of the lowest order desired in the product. For example, if the product is to be correct to hundredths, place the units' digit of the multiplier under thousandths' place of the multiplicand.

2. Write the first written digit in each partial product under the units' digit of the multiplier.

3. Multiply each digit of the multiplier by the digit directly above it in the multiplicand, first allowing for anything "carried" in multiplying the digit on the right in the multiplicand.

Ex. Multiply 345.2647 by 23.548, the answer to be correct to hundredths.

```
 345.2647
    84532
 --------
 6905.294
 1035.794
  172.632
   13.810
    2.762
 --------
 8130.292   8130.29 Ans.
```

In multiplying, for example, by the 4 in the above, we say four 6's are 24. Since 24 is nearer 20 than 30 there are 2 to carry. Four 2's are 8 and 2 (carried) are 10. Write the 0 under the 2 of the third partial product and carry the 1. Four 5's are 20 and 1 (carried) are 21; etc.

Working for a specified degree of accuracy in division. Much work may be saved also in the division of decimals

where a specified degree of accuracy is known in advance. The method is as follows:

1. Determine the number of digits in the final answer and retain one more than this number in the divisor, counting from the left.

2. After each successive digit is written in the quotient and the multiplication is performed, cut off a digit from the right of the divisor instead of "bringing down" a digit of the dividend.

3. In the next multiplication consider the digit that was cut off in order to get the number carried.

Ex. Divide 3.182 by 14.62, the final answer to be correct to hundredths.

$$\begin{array}{r} \cdot\;\; \\ 14.6\cancel{2})\overline{3.182} \end{array}$$

We find by inspection that the first significant digit of the quotient is in tenths' place. Since the final answer is to be correct to hundredths, it must then contain two digits. Hence we retain three digits in the divisor, counting from the left.

$$\begin{array}{r} .2\;\; \\ 14.6\cancel{2})\overline{3.182} \\ \underline{2\ 92} \\ 26 \end{array}$$

We say two 2's are 4. There is nothing to carry since 4 is less than 5. After getting the remainder 26, we cut off the 6 in the divisor.

$$\begin{array}{r} .21\; \\ 14.\cancel{6}\cancel{2})\overline{3.182} \\ \underline{2\ 92} \\ 26 \\ \underline{15} \end{array}$$

We must now divide 14 into 26. It is contained once. We first say once 6 is 6, making 1 to carry since 6 is nearer 10 than 0. Multiplying 14 by 1 and adding the 1 (carried) gives 15 to subtract. The completed solution stands:

$$\begin{array}{r} .218 \\ 1\cancel{4}.\cancel{6}\cancel{2})\overline{3.182} \\ \underline{2\ 92} \\ 26 \\ \underline{15} \\ 11 \\ \underline{11} \end{array} \qquad .22\ \textit{Ans.}$$

Ex. Divide 3.182 by .1462, making the final answer correct to hundredths.

```
           21.765      21.76 Ans.
.14620)3.1820
       2 9240
         2580
         1462
         1118
         1023
           95
           88
            7
            7
```

Where the last quotient figure is 5, in this approximate method it may be more nearly exact to "take" than to "give" as in the above answer.

Ex. Divide 18.2 by 3.1416, the final answer to be correct to tenths. The following solution omits the partial products.

```
            5.79      5.8 Ans.
3.1416)18.20
        2 49
          29
           1
```

PERCENTAGE

Fundamental ideas. While percentage does not add anything to our information regarding the fundamental operations in arithmetic, the symbolic use of fractions as employed in percentage plays an important part in computation. In percentage we treat fractions, both common and decimal, as having a denominator 100.

The words "hundredths" (or hundredth) and "per cent" are interchangeable. Thus 7 per cent means 7 hundredths. The latter is written $\frac{7}{100}$ or .07, the former *7%*.

DECIMALS TO PER CENTS

Before the pupil can fully understand how to change decimals to per cents he should be able to change decimals to hundredths. The steps leading from decimals to per cents are illustrated in the following examples:

$$.37 = 37 \text{ hundredths} = 37\%$$
$$.375 = 37\tfrac{1}{2} \text{ hundredths} = 37\tfrac{1}{2}\%$$
$$.0375 = 3\tfrac{3}{4} \text{ hundredths} = 3\tfrac{3}{4}\%$$
$$3.7 = 370 \text{ hundredths} = 370\%$$
$$.0015 = \tfrac{15}{100} \text{ of 1 hundredth} = \tfrac{3}{20} \text{ of } 1\%$$

The middle steps in the above may soon be omitted, the rule being observed that when the symbol % is used the point is moved two places to the right.

Always say $\frac{3}{4}$ of 1%, not $\frac{3}{4}$%, as the former is the proper business usage and conveys a definite meaning.

PER CENTS TO DECIMALS

In changing from per cents to decimals, reverse the steps in examples like those illustrated above.

$$17\% = 17 \text{ hundredths} = .17$$
$$6\tfrac{2}{3}\% = 6\tfrac{2}{3} \text{ hundredths} = .06\tfrac{2}{3}$$
$$37\tfrac{1}{2}\% = 37\tfrac{1}{2} \text{ hundredths} = .37\tfrac{1}{2} = .375$$
$$\tfrac{1}{2} \text{ of } 1\% = \tfrac{1}{2} \text{ of 1 hundredth} = .00\tfrac{1}{2} = .005$$
$$125\% = 125 \text{ hundredths} = 1.25$$

The middle steps are not needed in practice. Deduce the rule, "To change per cents to decimals, omit the symbol for per cent and point off two places to the left."

COMMON FRACTIONS TO PER CENTS

Use of the aliquot parts. Make use of the aliquot parts, previously memorized, as in the following:

$$\tfrac{1}{5} = .20 = 20\%$$
$$\tfrac{3}{8} = .37\tfrac{1}{2} = 37\tfrac{1}{2}\%$$
$$\tfrac{1}{3} = .33\tfrac{1}{3} = 33\tfrac{1}{3}\%$$
$$1\tfrac{1}{2} = 1.50 = 150\%$$

The middle steps are omitted in practice, for the class should immediately memorize all the aliquot parts and their per cent equivalents. The circle and its equal subdivisions may aid here.

We may reduce $\frac{3}{5}$ to per cent either by stating that since $\frac{1}{5} = 20\%$, $\frac{3}{5} = 3 \times 20\% = 60\%$; or by remembering that $\frac{3}{5}$ equals .60, which equals 60%. It is perhaps better to use the first method, but much depends upon the given fraction. Many fractions can be changed to per cents by relating them to certain fractions whose per cent equivalents are known.

Ex. Knowing that $\frac{1}{6} = 16\frac{2}{3}\%$, change $\frac{5}{6}$ to per cent.

The fundamental idea in changing fractions, both common and decimal, to per cents is to change them to hundredths. We have already shown this in connection with decimals, on p. 102, where we deduced the rule for changing decimals to per cents. Immediately above we have shown that certain familiar common fractions and their multiples can be changed mentally into decimals and then written as per cents. How shall we change the less familiar common fractions to per cents?

Less familiar fractions. They may be changed as follows:

(*a*) Change to hundredths, retaining the common fraction form.

Exs. $\frac{1}{50} = \frac{2}{100} = 2\%$
$\frac{1}{500} = \frac{1}{5}$ of $\frac{1}{100} = \frac{1}{5}$ of 1%
$\frac{2}{800} = \frac{2}{8}$ of $\frac{1}{100} = \frac{2}{8}$ of 1%
$\frac{1}{125} = \frac{8}{1000} = \frac{4}{5}$ of $\frac{1}{100} = \frac{4}{5}$ of 1%

This method is good where the denominator is a factor of 100 or 1000, or is some multiple of 100. In other cases use the following method:

(*b*) Change to a decimal fraction, expressed as hundredths, by dividing the numerator by the denominator, as in the division of decimals.

Ex. $\frac{3}{13} = .23\frac{1}{13} = 23\frac{1}{13}\%$. SIDE WORK: $13\,)\underline{3.00}$
$.23\frac{1}{13}$

It may be preferred by some to reduce fractions in method (*a*) by this method, even when the division is exact.

Exs. $\frac{1}{50} = .02 = 2\%$. SIDE WORK: $50\,)\underline{1.00}$
$.02$

$\frac{1}{125} = .008 = .00\frac{8}{10} = .00\frac{4}{5} = \frac{4}{5}$ of 1%.
$.008$
SIDE WORK: $125\,)\overline{1.000}$
$\underline{1\ 000}$

It should be noted, however, that these examples are worked more simply by method (*a*). Reduce $\frac{2}{800}$ to per cent by method (*b*) and observe the complexity that is liable to arise. Aim to choose the method that best fits the example.

Bases for reference. Do not neglect to teach that once a number is 100% of it, twice a number is 200% of it, etc. That is, 1 = 100%, 2 = 200%, etc. Select 100%, 200%,

etc., as bases for reference in reducing mixed numbers to per cents:

$$2\tfrac{1}{5} = 200\,\% + 20\% = 220\,\%.$$
$$1.15 = 100\% + 15\,\% = 115\,\%,$$

or use the rule to move the point two places to the right and use the symbol %.

Very small fractions and mixed numbers are apt to be confusing. Choose 1 % as a basis for reference in the former:

$\frac{1}{200} = \frac{1}{2}$ of $\frac{1}{100} = \frac{1}{2}$ of 1 %. Or,

$\frac{1}{200} = 1 \div 200 = .005 = .00\tfrac{1}{2} = \frac{1}{2}$ of 1 %. SIDE WORK: $200\overline{)1.0}$
$.005$

PER CENTS TO COMMON FRACTIONS

Use of the aliquot parts. In changing from per cents to common fractions make use of any of the aliquot parts memorized when changing from common fractions to per cents. It is not necessary, for instance, to develop the fact that $33\tfrac{1}{3}\,\% = \tfrac{1}{3}$, $37\tfrac{1}{2}\,\% = \tfrac{3}{8}$, $60\% = \tfrac{3}{5}$, etc.

Less familiar per cents. In changing less familiar per cent values to common fractions, proceed as in the examples:

$$28\,\% = 28 \text{ hundredths} = \tfrac{28}{100} = \tfrac{7}{25}$$
$$3\tfrac{3}{4}\,\% = 3\tfrac{3}{4} \text{ hundredths} = \quad = \tfrac{15}{400}$$
$$\tfrac{3}{4} \text{ of } 1\,\% = \tfrac{3}{4} \text{ of } 1 \text{ hundredth} = \quad = \tfrac{3}{400}$$

Fill in the above omissions. Always reduce answers to lowest terms. The second steps in the above are omitted in practice.

It may sometimes be preferable to change first to the decimal form: $246\,\% = 2.46 = 2\tfrac{46}{100} = 2\tfrac{23}{50}$. But throughout reduction work in percentage do not express the same fraction in both the common and the decimal form unless

there is something to be gained. There is in the following example a very common method of changing common fractions to per cents that should be mentioned. Such serious errors are apt to arise from its use, however, that we hesitate to give it.

Ex. Change $\frac{3}{7}$ to per cent.
SOLUTION: (*a*) $\frac{3}{7} \times 100\% = \frac{300}{7}\% = 42\frac{6}{7}\%$.

NOTE. — Pupils are liable to omit one or more of the per cent signs and hence to write meaningless statements of equality. Where fundamental ideas are needed, as they are in percentage, there should be no opportunity for misconceptions. Remembering the rule for changing decimals to per cents, we may apply the principle of that rule here, that is, multiply by 100 and then affix the symbol %. We write: (*b*) $\frac{3}{7} \times 100 = \frac{300}{7} = 42\frac{6}{7}$; $42\frac{6}{7}\%$. *Ans.* This also is correct, but it does not lend itself to sound treatment by the pupils.

If the teacher in using this method can get the class to write statements that mean something, as is true in solutions (*a*) and (*b*) above, then either of these forms may be used.

The explanation of solution (*a*) is: Once anything is 100% of it. Then $\frac{3}{7}$ of the same thing is $\frac{3}{7}$ of 100% of it, or as is usually written, $\frac{3}{7} \times 100\%$, which equals $42\frac{6}{7}\%$. A circle representing 100% could be used to help make this clear.

QUESTIONS

1. Change to per cents: .035, $\frac{5}{7}$, $\frac{1}{250}$, $3\frac{1}{8}$, .007.
2. Change to common fractions: $4\frac{1}{2}\%$, $\frac{3}{4}$ of 1%, $262\frac{1}{2}\%$.
3. Change to decimal fractions: $\frac{1}{8}$ of 1%, $29\frac{1}{2}\%$, 384%, 1875%.
4. Simplify $\dfrac{66\frac{2}{3}\% \text{ of } 1800 \times .00625}{\frac{1}{8} \text{ of } 1\% \text{ of } 2000}$.

Solution: The above equals $\dfrac{2 \times 1800 \times 5}{3 \times \frac{1}{800} \times 2000 \times 800}$

$$= \frac{2 \times 1800 \times 5}{3 \times 4 \times 800} = 1\tfrac{7}{8}.$$

5. Simplify $\dfrac{240 \times \frac{3}{8} \text{ of } 1\% \text{ of } 1500 \times .0025}{.003 \times .37\frac{1}{2} \times 66\frac{2}{3}}$.

TYPE EXAMPLES IN FRACTIONS AND IN PERCENTAGE

Examples *vs.* problems. The examples to be considered in this connection are not problems, strictly speaking, but merely questions or exercises concerning relations between abstract numbers, integral and fractional. The term "problem" relates to exercises having concreteness in relation to commodities and to daily experience.

Corresponding types in whole numbers and percentage. Three simple measure-content relations form the basis for three distinctive types of examples that occur in fractions and in percentage. These types have their earliest associations with whole numbers. In considering the equation $2 \times 4 = 8$, three different examples will arise if we substitute in turn a question mark or the algebraic x for each of the numbers. These examples are:

(*a*) What is 2×4?	or $2 \times 4 = x$.
(*b*) 8 is how many times 4?	or $x \times 4 = 8$.
(*c*) 8 is 2 times what number?	or $2 \times x = 8$.

The equivalents of these in percentage are:

(*a*) What is 200% of 4?	or 200% of $4 = x$.
(*b*) 8 is what per cent of 4?	or $x \times 4 = 8$.
(*c*) 8 is 200% of what number?	or 200% of $x = 8$.

Corresponding types in fractions and percentage. In considering the product $\frac{1}{2}$ of $8 = 4$, we have the same corresponding types. In fractions we have:

(*a*) What is $\frac{1}{2}$ of 8?	or $\frac{1}{2}$ of $8 = x$.
(*b*) 4 is what part of 8?	or $x \times 8 = 4$.
(*c*) 4 is $\frac{1}{2}$ of what number?	or $\frac{1}{2}$ of $x = 4$.

The equivalents of these in percentage are:

(*a*)	What is 50% of 8?	or	50% of $8 = x$.
(*b*)	4 is what per cent of 8?	or	$x \times 8 = 4$.
(*c*)	4 is 50% of what number?	or	50% of $x = 4$.

The corresponding cases in percentage. These three types we may call respectively Case I, Case II, and Case III. In percentage the multiplicand is called the base, the multiplier the rate, and the product the percentage. All problems in percentage are variations of these three types. In this segregation of percentage examples into distinctive classes, it must be understood by the reader that the purpose is to make our treatment more definite. In schoolroom practice it is not advisable to train pupils to work by cases, each case illustrated by the solution of a type example, which solution the pupils duplicate in solving other examples under that case. On the other hand the teacher cannot expect good results when assuming that the pupils will naturally solve the various classes of examples without guidance and drill. There should be a proper balance between these two extremes.

Variations of the types. The following variations of these types are given to illustrate the use of such phrases as *times*, *part of*, *times greater*, *part greater*, *part less*, *per cent of*, *per cent greater*, and *per cent less*. The right side of the page illustrates the percentage phraseology, the left side that of common fractions. The numbers or words within most of the parentheses are commonly omitted, but are inserted here to convey the true meaning of the example.

Case I

Given the base and rate per cent, to find the percentage.

1

(*a*) *Times.*
What is 3 times 16?

(*b*) *Part of.*
What is $\frac{1}{4}$ of 12?

(*a*) *Per cent of.*
What is 300 % of 16?

(*b*) *Per cent of.*
What is 25 % of 12?

2

(*a*) *Times greater than.*
What number is three times (16) greater than 16?

(*b*) *Part of greater than.*
What number is $\frac{1}{4}$ (of 12) greater than 12?

(*c*) *Part of less than.*
What number is $\frac{1}{4}$ (of 12) less than 12?

(*a*) *Per cent of greater than*
What number is 300 % (of 16) greater than 16?

(*b*) *Per cent of greater than.*
What number is 25 % (of 12) greater than 12?

(*c*) *Per cent of less than.*
What number is 25 % (of 12) less than 12?

Case II

Given the base and percentage, to find the rate per cent.

1

(*a*) *Times.*
48 is how many times 16?

(*b*) *Part of.*
3 is what part of 12?

(*a*) *Per cent of.*
48 is what per cent of 16?

(*b*) *Per cent of.*
3 is what per cent of 12?

2

(*a*) *Times greater than.*
64 is how many times (16) greater than 16?

(*b*) *Part of greater than.*
15 is what part (of 12) greater than 12?

(*c*) *Part of less than.*
9 is what part (of 12) less than 12?

(*a*) *Per cent of greater than.*
64 is what per cent (of 16) greater than 16?

(*b*) *Per cent of greater than.*
15 is what per cent (of 12) greater than 12?

(*c*) *Per cent of less than.*
9 is what per cent (of 12) less than 12?

Case III

Given the rate per cent and percentage, to find the base.

1

(*a*) *Times.*
48 is 3 times what number?

(*b*) *Part of.*
3 is $\frac{1}{4}$ of what number?

(*a*) *Per cent of.*
48 is 300% of what number?

(*b*) *Per cent of.*
3 is 25% of what number?

2

(*a*) *Times greater than.*
64 is 3 times (what number) greater than what (that) number?

(*b*) *Part of greater than.*
15 is $\frac{1}{4}$ (of what number) greater than what (that) number?

(*a*) *Per cent of greater than.*
64 is 300% (of what number) greater than what (that) number?

(*b*) *Per cent of greater than.*
15 is 25% (of what number) greater than what (that) number?

(*c*) *Part of less than.*
9 is $\frac{1}{4}$ (of what number) less than what (that) number?

(*c*) *Per cent of less than.*
9 is 25% (of what number) less than what (that) number?

When the aliquot-part relations can be used to advantage in solving percentage examples always convert them into their equivalents, as given on the left side of the page in the above.

SOLUTION OF EXAMPLES IN FRACTIONS AND IN PERCENTAGE

Value of the equation. How are pupils to tell under which case different examples come? The best answer is that if they are trained to make use of the equation, there is little need of remembering cases. It is true, however, that the equation is a hindrance where the examples are so simple that they can be solved mentally. In the following treatment of the cases both of these aspects are kept in mind.

UNDER CASE I

Examples with one step. Pupils early understand how to get $\frac{1}{3}$ of 180, hence when they are asked to get $.33\frac{1}{3}$ of 180, or, later, $33\frac{1}{3}$% of 180, they reduce the work to the first form. They learn how to get .32 of 66 in the multiplication of decimals, hence they have a basis for finding 32% of 66. Sometimes it is convenient to find first 1% of a given number, as in the following solution:

Ex. Find $\frac{1}{2}$% ($\frac{1}{2}$ of 1%) of 380.

SOLUTION: First get 1% of 380 by moving the point two places to the left. The necessary written work is: $2\,\overline{)3.80}$

1.90 *Ans.*

The equation that may be used in solving examples like the above is $p = r \times B$, in which r is the known rate; B, the base, the number to be multiplied; and p, the percentage, the product.

We have $p = r \times B$. Substituting the known values, we get

$$p = 33\tfrac{1}{3}\% \text{ of } 180, \text{ where } r = 33\tfrac{1}{3}\% \text{ and } B = 180$$
$$p = \tfrac{1}{3} \text{ of } 180$$
$$p = 60.$$

Examples with two steps. In the following example two steps are involved:

Ex. What number is 25% greater than 12?

SOLUTIONS: (*a*) First find 25% of 12. It is $\frac{1}{4}$ of 12, or 3. Add 3 to 12, giving 15. *Ans.*

(*b*) Get 125% of 12. It is $1\frac{1}{4} \times 12$, or 15. *Ans.*

The equation form is:

(*a*) $p = r \times B$	Or (*b*)	$P = R \times B$
$p = 25\%$ of 12		$P = 125\%$ of 12
$p = \frac{1}{4}$ of 12		$P = \frac{5}{4}$ of 12
$p = 3$		$P = 15$.
$3 + 12 = 15$. *Ans.*		P is written instead of p to correspond to R, which here represents $100\% + r$.

UNDER CASE II

Examples with one step. Under the second case, the rate per cent, part of, or times, is required. The pupil gets his basis for the solution from simple examples in early work. He learns, for example, that $\frac{1}{2}$ of 8 is 4 and, in close connection, that the part 4 is of 8 is $\frac{1}{2}$, obtained by dividing 4 by 8 in the fraction form and reducing. He may also be trained to say in full that 4 is $\frac{4}{8}$ of 8, or $\frac{1}{2}$ of 8. It may aid in quick mental work to remember that the number following "of" (or times) should be taken as the denominator (or divisor). Thus 18 is what part of 24? $\frac{18}{24} = \frac{3}{4}$. *Ans.* Also 18 is how many times 6? $\frac{18}{6} = 3$. *Ans.*

If the rate per cent is required, proceed as above and reduce the fraction to per cent.

Ex. 4 is what per cent of 12? The corresponding example in fractions is: 4 is what part of 12? 4 is $\frac{1}{3}$ of 12. $\frac{1}{3} = 33\frac{1}{3}\%$. Therefore 4 is $33\frac{1}{3}\%$ of 12.

Ex. 16 is what per cent of 4? Restated, we have: 16 is how many times 4? 16 is 4 times 4. Four times a number is 400% of it. Therefore 16 is 400% of 4.

The equation is used as follows:

Ex. 4 is what per cent of 12?

$$r \times B = p$$
$$r \times 12 = 4$$
$$r = \tfrac{4}{12}$$
$$r = \tfrac{1}{3}, \text{ or } 33\tfrac{1}{3}\%.$$

Ex. 16 is what per cent of 4?

$$r \times B = p$$
$$r \times 4 = 16$$
$$r = \tfrac{16}{4} = 4, \text{ or } 400\%.$$

Examples with two steps. Where two steps are involved in this case, proceed as in the following solutions:

Ex. 20 is what per cent greater than 18? Where the equation is not used, either of two methods may be employed.

(*a*) First find how much greater 20 is than 18. It is 2 greater. Next find what part 2 is of 18. It is $\frac{2}{18}$, or $\frac{1}{9}$, of 18. $\frac{1}{9} = 11\frac{1}{9}\%$. *Ans.*

(*b*) First find how many times 18 is 20. It is $\frac{20}{18}$ of 18, or $1\frac{1}{9}$ times 18. Therefore 18 is $\frac{1}{9}$, or $11\frac{1}{9}\%$, greater than 18.

By using the equation we may, as in (*a*) above, find that 20 is 2 greater than 18, and then solve for r in $r \times 18 = 2$. Or we may immediately find the per cent 20 is of 18 as in the following solution:

$$R \times B = P$$
$$R \times 18 = 20$$
$$R = \tfrac{20}{18} = \tfrac{10}{9} = 1\tfrac{1}{9} = 111\tfrac{1}{9}\%$$
$$r = 111\tfrac{1}{9}\% - 100\% = 11\tfrac{1}{9}\%. \textit{ Ans.}$$

Ex. Find mentally the per cent greater than or less than that the first number is of the second in the following: 18, 20; 24, 20; 20, 24; 4, 6; 6, 4.

Ex. Using the equation, find the per cent greater or less the first number is of the second in the following: 125, 140; $1\frac{1}{3}$, 1.25.

Drill in solving equations. If the equation is used, it will be necessary to have drill in solving for the unknown quantity in an equation like $r \times B = p$ without any reference to fraction or percentage examples. Assign values to two of the letters and solve for the third. Do the same in an equation involving other letters, like $a \times b = c$.

Ex. In $a \times b = c$, solve for the third letter where $a = 2$, $c = 3$; $a = \frac{1}{2}$, $b = 5$; $b = \frac{3}{4}$, $c = 5$.

In solving for a in $a \times \frac{5}{6} = 20$, write $a = 20 \div \frac{5}{6} = 20 \times \frac{6}{5} = 24$.

Such work will give the class the necessary preparation for work like that already given and that which is to follow in examples under Case III.

UNDER CASE III

Examples with one step. Under the third case the product and the multiplier are given and it is required to find the multiplicand, or, as it is expressed in percentage, the percentage and the rate per cent are given and it is required to find the base. When the pupil learns that $\frac{1}{3}$ of 12 is 4, he can remember the number of which 4 is $\frac{1}{3}$, but he needs a method of finding the number of which 18 is $\frac{3}{5}$. The equation is especially helpful in this connection:

We have

$$r \times B = p$$
$$\tfrac{3}{5} \times B = 18$$
$$B = 18 \div \tfrac{3}{5} = 18 \times \tfrac{5}{3} = 30.$$

"This is also the solution of 18 is 60% of what number?"

Ex. 22 is $27\frac{1}{2}\%$ of what number?

$$r \times B = p$$
$$.275 \times B = 22$$
$$B = \frac{22}{.275} = 80.$$

Unitary analysis. The algebraic method illustrated above has, to a great extent, been substituted for unitary analysis, which is illustrated as follows:

Ex. 18 is $\frac{3}{5}$ of what number?

$\frac{3}{5}$ of the number $= 18$
$\frac{1}{5}$ of the number $= \frac{1}{3}$ of 18, or 6
$\frac{5}{5}$ of the number $= 5 \times 6$, or 30
Therefore the number $= 30$.

If we use B, n, or x for the phrase "the number" and divide by $\frac{3}{5}$ at once instead of first finding $\frac{1}{5}$ of the number, we get what has been used above.

Ex. 18 is 60% of what number?

60% of the number $= 18$
1% of the number $= \frac{1}{60}$ of 18, or $\frac{3}{10}$
100% of the number $= 100 \times \frac{3}{10}$, or 30
Therefore the number $= 30$.

Ex. 22 is $27\frac{1}{2}\%$ of what number?

$27\frac{1}{2}\%$ of the number $= 22$
$$1\% \text{ of the number} = \frac{1}{27\frac{1}{2}} \text{ of } 22, \text{ etc.}$$

The difficulty that appears in the above reasoning can be avoided by changing $27\frac{1}{2}\%$ either to a common fraction or to a pure decimal:

$$27\tfrac{1}{2}\% = \tfrac{27.5}{100} = \tfrac{11}{40}.$$

Then $\frac{11}{40}$ of the number $= 22$
$\frac{1}{40}$ of the number $= \frac{1}{11}$ of 22, etc.

Or .275 of the number $= 22$
.001 of the number $= \frac{1}{275}$ of 22
$\frac{1000}{1000}$ of the number $= 1000 \times \frac{1}{275}$ of 22, etc.

The last three solutions are, of course, not recommended. The use of the algebraic equation as shown above presents no such difficulties.

Examples with two steps. Examples with two steps in the solutions are illustrated in the following:

Ex. 25 is $16\frac{2}{3}\%$ less than what number?

$16\frac{2}{3}\% = \frac{1}{6}$. The statement that 25 is $\frac{1}{6}$ less than a number is equivalent to the statement that 25 is $\frac{5}{6}$ of a number. The example is thus reduced to one of the types discussed above.

We get
$$R \times B = P \quad (R = 100\% - r)$$
$$\tfrac{5}{6} \times B = 25$$
$$B = 25 \div \tfrac{5}{6} = 25 \times \tfrac{6}{5} = 30.$$

Ex. 44.8 is 12% greater than what number?
Restated, we have 44.8 is 112% of what number?
$$112\% \text{ of } B = 44.8$$
$$\text{or } 1.12 \times B = 44.8$$
$$B = \frac{44.8}{1.12} = 40.$$

Exs. Solve mentally: 20 is $33\frac{1}{3}\%$ greater than what number? 15 is $16\frac{2}{3}\%$ less than what number? 30 is 20% of what number?

Exs. Solve by using the equation: 6600 is $66\frac{2}{3}\%$ greater than what number? 143.52 is 8% less than what number?

SQUARE ROOT

By inspection. Before learning the standard form of extracting square root, pupils should extract square root and cube root by inspection. The extraction of cube root by the long form is not necessary, as it is rarely used in everyday life, but it is advisable to have various roots found by inspection in order to bring out the meaning of roots and of square root in particular.

In extracting square root by inspection try to find the largest possible factor that is a perfect square.

Ex. Find the square root of 576.

$$576 = 9 \times 64,$$

Hence $\sqrt{576} = \sqrt{9 \times 64} = 3 \times 8 = 24.$

Ex. $\sqrt{196} = \sqrt{4 \times 49} = 2 \times 7 = 14.$

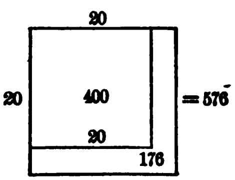
Ex. $\sqrt{135} = \sqrt{9 \times 15}$. Since the first factor is a perfect square and the second one is not, the square root of 135 cannot be found exactly.

If it is not convenient to find a factor that is a perfect square, factor the number into its prime factors and arrange the factors into two like groups, thus

$$576 = (3 \times 2 \times 2 \times 2) \times (3 \times 2 \times 2 \times 2)$$
$$\sqrt{576} = 3 \times 2 \times 2 \times 2 = 24.$$

By the standard method. In case the class has had sufficient algebra, apply to arithmetic the algebraic rule for extracting square root. Otherwise give the rule as commonly given in arithmetic, using a diagram to help fix ideas.

Explain the reason for pointing off the figures from the decimal point in groups of twos. This is due to the fact that when a number is squared the product will have either twice the number of figures in the number squared or twice the number less 1. Write a list of squares to show the truth of this.

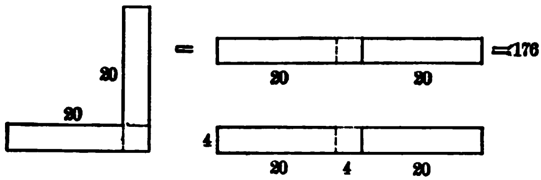


Ex. Extract the square root of 576.

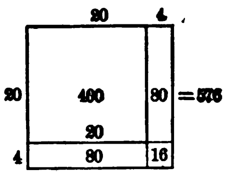
$$\begin{array}{rl} & 5'76)\underline{20} \\ 20^2 = & \underline{4\ 00} \\ & 1\ 76 \end{array}$$

The largest square in 576 is 400, its side being 20. Subtracting 400 from 576, we get 176 as a remainder. The L-shaped part of the figure is the remainder. We wish to find the width of

this strip. This being determined, we can add it to 20, and get the width of the given square, that is, the square root of 576. An approximate width is found by dividing the area of the strip, 176, by the approximate length, which is 2×20. $176 \div 40 = 4$, the approximate width.



The length of the rectangle must be increased by 4 to get a closer approximation. Multiplying the new approximate length of the rectangle, $2 \times 20 + 4$, by the approximate width 4, we get exactly 176. Hence 4 is the exact width of the rectangle, and the width of the original square is $20 + 4 = 24$. That is, the square root of 576 is 24.



$$\begin{array}{rl} & 5'76)\underline{20 + 4} \\ 20^2 = & \underline{4\ 00} \\ 2 \times 20 = 40 & 1\ 76 \\ (40 + 4)4 = & \underline{1\ 76} \end{array}$$

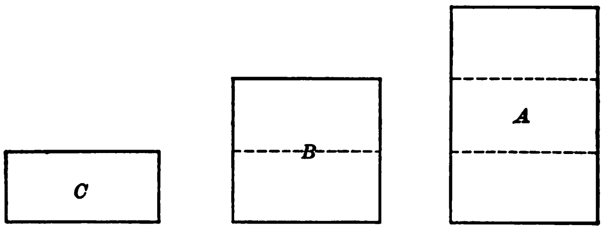
A good final form is as follows, where mental work permits of a minimum of work on the left. In longer examples it may be found advisable to retain the full written side work.

$$\begin{array}{rl} & 5'76)\underline{24} \\ & \underline{4} \\ & 1\ 76 \\ 44 & \underline{1\ 76} \end{array}$$

Ex. Extract the square root of 60,625, using a diagram for illustrative purposes. After subtracting the first square, 40,000, the approximate width of the L-shaped strip is found to be 40. Observe that there finally remains a narrower strip whose area corresponds to the second remainder in the solution. The width of this strip is found to be 5.

RATIO AND PROPORTION

Early idea of ratio. Pupils learn of ratio in the first year while studying comparative magnitudes, although the term "ratio" is not generally used so early. They learn, for example, that object *A* is 3 times object *C*, and that object *C* is $\frac{1}{3}$ of object *A*. As fraction work becomes more prominent in the earlier grades, they learn that object *B* is $\frac{2}{3}$ of object *A*, and that object *A* is $1\frac{1}{2}$ times object B.



In the language of ratio they eventually learn that the ratio of *B* to *A* is as 2 to 3, written $\frac{2}{3}$ or 2 : 3. The idea of ratio is used under the phrases, "how many times," "what part of," or expressed as "a relation between." The latter phrase is an indefinite expression, and should be replaced by "ratio between," which is defined under the first two phrases.

The use of ratio. In practice we do not always want to know exactly what part one magnitude or number is of another or how many times another, but are satisfied to know instead the least whole numbers that we may compare. For instance, the ratio of the population of two cities is found to be as 5 to 7, and we have an easy basis for comparison. The laws that apply to fractions apply to ratios.

The pupil must learn the language of ratio and after that he needs only to employ principles learned in fractions.

Written forms in ratio and proportion. Written forms in ratio and proportion are in a period of transition, historically speaking. In ratio, the form 2 : 3 is less commonly used than formerly, being replaced by the more definite expression $\frac{2}{3}$. In like manner in proportion the form $\frac{2}{3} = \frac{8}{x}$ is coming more into common use. The form $2 : 3 = 8 : x$ is preferred by many teachers, but the use of the four dots, as in $2 : 3 :: 8 : x$, is going out of use.

Essential principles in proportion. Pupils in the elementary school have little occasion to know the theory of proportion. They must know first that a proportion is a statement of equality between two ratios. Beyond this it is sufficient that they know how to build up a proportion from the data of a problem and that they can solve for the unknown term. While it seems natural to select the fourth term for the unknown, this is probably a matter of tradition.

It seems wise to use the form $\frac{2}{3} = \frac{8}{x}$, for in the solution the pupil learns the algebraic principles employed in clearing of fractions and solving for x and adds something to his understanding of the practical applications of algebra. The old method of using a question mark in place of the desired term and stating a rule for the solution had little to commend it. For example, in solving $2 : 3 = 8 : (?)$, the rule was to multiply the two means and divide the product by the given extreme to find the other extreme.

Value of proportion. The use of proportion adds much to facility in computation. It is therefore to be regretted that texts and teachers give the subject but a passing notice, except perhaps where it is employed in relation to similar figures. The method of proportion may be employed informally in the solution of the problem, (*a*) "If 4 tiers of wood cost $10, how much will 12 tiers cost?" where we determine that 12 tiers will cost 3 times the cost of 4 tiers. Again in the problem, (*b*) "If 2 boys can pile some wood in 12 hours, how many hours will it take 8 boys to pile the same amount?" we conclude that it will take 8 boys $\frac{1}{4}$ as many hours as it takes 2 boys. Solutions like these embody a fundamental understanding of proportion, although no formal proportion has been written down. Pupils who can work problems like the above only by first finding the cost of 1 tier or the number of hours it takes 1 boy lose something of ease and facility in computation.

Implied element in a proportion. It is not surprising that pupils occasionally find difficulty in writing proportions. It is perhaps due to the absence of an element whose existence is implied in the problem. Thus in the first example above it is implied that 1 tier of wood costs the same in each case. In the second example it is implied that all of the boys in both groups (8 boys and 2 boys) work at the same speed. One's faith in proportion may be made secure after the following considerations:

In problem (*a*) above, let y equal the cost of 1 tier. Then we have the two equations:

$$4y = 10$$
$$12y = x.$$

Dividing the members of the first equation by the corresponding members of the second, we obtain

$$\frac{4}{12} = \frac{10}{x}, \text{ or } 4:12 = 10:x.$$

In problem (*b*) above, let *y* equal the number of hours it takes one boy alone to do the work. We have the two equations:

$$\frac{y}{8} = x$$

$$\frac{y}{2} = 12.$$

Dividing corresponding members, we obtain

$$\frac{y}{8} \div \frac{y}{2} = \frac{x}{12}.$$

Or $$\frac{2}{8} = \frac{x}{12}, \text{ or } 2:8 = x:12.$$

Terms directly and inversely proportional. The terms in problem (*a*) above are said to be directly proportional, while those in problem (*b*) are inversely proportional, inversely because the data of the third and fourth terms correspond respectively to the second and first terms and not to the first and second.

State which of the following are directly and which inversely proportional and what factor is constant in each:

1. Cost of transporting goods; the distance carried.

Ans. Directly proportional, for the greater the distance goods are carried, the greater the cost. The constant factor is the rate (not constant in practice).

2. The price of flour; the size of a loaf of bread.
3. The price of flour; the price of a loaf of bread.
4. The size of a loaf of bread; the weight of a loaf.

Ex. If when flour is $1.25 a sack the weight of a loaf of bread is 8 oz., what should be the weight of a loaf when flour is $1.75 a sack?

Ex. If when flour is $1.25 a sack the price of a loaf of bread is 5 cents, what should be the price of a loaf when flour is $1.75 a sack?

Comparison of like concrete terms. Like terms should be compared in proportion. That is, the first two terms should be alike in quality, and also the last two. Thus we write 4 hats : 6 hats = $12 : $18, but not 4 hats : $12 = 6 hats : $18. In dealing with abstract numbers we are free, of course, to write either 4 : 6 = 12 : 18 or 4 : 12 = 6 : 18.

PLAN OF EXPLANATION OF MECHANICAL PROCESSES

Early language forms. Beginning in the first grade, children are trained to give oral expression to their ideas. This is illustrated in story problems and in descriptions of "discoveries." The use of language forms in connection with the early mechanical processes is highly desirable. The following form may be used when pupils are first learning to add two columns:

$$\begin{array}{r} 42 \\ 23 \\ \underline{29} \\ 94 \end{array}$$

The pupil says when adding the first column, 12, 14. Write down the 4 and add the 1 to the next lowest figure. 3, 5, 9. Write down the 9. The sum of the numbers is 94.

The following language form may be used in connection with the hard case in subtraction until the results show that the process will not be readily forgotten:

$$\begin{array}{r} 92 \\ -\underline{33} \\ 59 \end{array}$$

Since 3 is greater than 2, no number added to 3 can give 2. Hence we think 12 in place of the 2. 3 and 9 are 12. Write down the 9 and add 1 to the next lower figure. 4 and 5 are 9. Write down the 5. The difference between 92 and 33 is 59.

Aims. The chief purpose in requiring pupils to explain their work orally is to aid the class in keeping the steps of the solutions clearly in mind. A secondary aim is involved in cultivating habits of systematic thinking in arithmetic on the part of the one explaining. This latter aim becomes a primary one where written work is concerned.

Time given to explanations. The portion of the recitation period that should be given to explaining depends upon the grade and upon the nature of the work. The chief aims of the recitation are to give the class facility in the mechanical processes and an appreciation for an understanding of their essential applications. The first of these, — facility, which implies drill, — is the more important. Sufficient explanation should be given to further the realization of these aims, but beyond this it is liable to be made a thing in itself, a sort of mental treadmill.

Plan of explanation. Two forms of explanation arise in class work, — that of stating the steps in mechanical processes and that of making clear the solution of problems. The general plan is the same in each.

Plan of explanation:

1. State the exercise. Ex. Multiply 2935 by 74.

2. Give the successive steps, stating, whenever practicable, what will be done, and then doing it. In the above example we proceed as follows:

(*a*) Multiply 2935 by 4. 4 times 5 is 20. Place the 0 under the 4; etc.

(*b*) Multiply 2935 by 7. 7 times 5 is 35. Place the 5 under the 7; etc.

(*c*) Add.

3. State the answer.

As the class becomes more familiar with the processes involved, the number of steps and the amount of detail should be correspondingly diminished.

The teacher should proceed systematically in this sort of drill in connection with the four fundamental operations, common and decimal fractions, square root, and wherever else any of these operations are employed, as in topics like factoring, greatest common divisor, least common multiple, denominate numbers, and mensuration.

CHAPTER V

THE APPLICATION SIDE OF ARITHMETIC — PERCENTAGE

SOLUTION OF PROBLEMS IN FRACTIONS AND IN PERCENTAGE

Plan of explanation. The same reasons hold for explaining content problems in class work as for explaining mechanical processes.[1] The same caution should be made also, that explanations cease to be of value when they become things in themselves. If used to clear up ideas and to establish orderly habits of thinking, they are productive of good.

Plan of explanation:

1. State the problem.

2. Give the explanation, resolving the main problem into a series of minor ones—in case these exist. In other words, explain by steps. Tell what is to be found in each step, whenever practicable, and then give the method of finding it.

3. State the answer in case it is not explicitly given in the last step of the explanation.

Illustrative explanations:

1. (1) Problem: If 4 books cost $12, how much will 9 books cost?
   (2) Explanation.
      (*a*) Find the cost of one book.
      Since 4 books cost $ 12, one book costs $\frac{1}{4}$ of $ 12, or $ 3.
      (*b*) Find the cost of 9 books.
      Since 1 book costs $ 3, nine books cost 9 times $3, or $ 27.

[1] See conclusion of preceding chapter.

2. (1) Problem: If 4 books cost $12, how much will 20 books cost?
   (2) Explanation.
      (*a*) 20 is 5 times 4.
      (*b*) Hence the cost of 20 books is 5 times the cost of 4 books.
      (*c*) The cost of 4 books is $12.
      (*d*) Therefore the cost of 20 books is 5 times $12, or $60.
3. (1) Problem: A lot was purchased for $900 and sold for $1200. What was the gain per cent?
   (2) Explanation.
      (*a*) Find the cash gain.
      The gain is the difference between the selling price, which is $1200, and the cost, which is $900, or $300.
      (*b*) Find what part $300, the gain, is of $900, the cost.
      It is $\frac{300}{900}$, or $\frac{1}{3}$.
      (*c*) Change $\frac{1}{3}$ to per cent.
      $\frac{1}{3} = 33\frac{1}{3}\%$.
   (3) Therefore the gain per cent is $33\frac{1}{3}\%$.

While the steps in the solution of simple problems like those above may be made less formal, the method is of special value where problems are more complex. The plan of telling what will next be done assists those listening to the explanation.

Several problems in one. The successive steps in the above problems are but a series of subordinate problems whose solutions in turn solve the main problem. In the primary grades children first learn to work problems with one step, or condition. Thus they learn (*a*) if 1 pencil costs 5 cents 6 pencils will cost 6×5 cents, or 30 cents. They learn later (*b*) if 4 pencils cost 20 cents, 1 pencil will cost $\frac{1}{4}$ of 20 cents, or 5 cents. Later they are given a problem which includes in its solution both these problems: If 4 pencils cost 20 cents, how much will 6 pencils cost?

It is the ability to resolve any problem into its subordi-

nate problems, or steps, and to determine which subordinate problem to solve first that brings success in solving the harder problems in arithmetic.

Problems with numbers omitted. The plan of working problems with some of the numbers omitted is an effective method of establishing principles. The method is illustrated in the following exercises, in which the first principles of proportion are developed:

1. If 4 books cost a certain amount, how much will 12 books cost? *Ans.* 3 times as much. How much will 2 books cost? 9 books?

2. If 4 men can do some work in a certain time, how long will it take 2 men? 12 men?

3. If the interest on a certain sum of money at 4% for a given time is $9.50, what is the interest at 8%? at 2%? at 6%?

4. If the interest on a certain sum of money at a given rate per cent for 2 months is $24, what is the interest for 8 months? for 1 month? for 5 months?

Drills like the above are particularly effective before working examples in which the full data are given. This principle of isolating certain features of a problem for independent drill is of value in all teaching, but especially so in arithmetic.

METHODS OF ATTACK

The problem as a whole. Problems involving addition and subtraction or addition and multiplication present little difficulty, but where both multiplications and divisions are involved, together with some possible additions and subtractions, trouble arises. An understanding of the direct processes, addition and multiplication, is the basis for an understanding of all the processes. Since the greatest difficulty lies in distinguishing when to multiply and when to

divide, we shall confine our discussion to these processes, which are the inverse of each other.

The first consideration in attacking a problem is to determine the subordinate problems the solutions of which will solve the main problem. The second consideration is to determine the order in which these problems, or steps, shall be solved. As an aid in accomplishing these two results the following plan is suggested:

Where possible divisions occur in a problem, recall the written form of the direct process (multiplication) and let this be a guide in the inverse process. To illustrate first by a simple example:

48	( )
3	3
144	144

Consider the product, 3 × 48. If the multiplicand were erased, we could find it by dividing 144 by 3, and should be solving the inverse example, 144 is 3 times what number?

In like manner consider the problems of which the accompanying mechanical work may be the solution. As the work stands it could be the solution of the problem:

```
     240
    1.15
    1200
   2640
  276.00
     .95
   1380
  2484
  262.20
```

"A man bought a lot for $240. He sold it again at a gain of 15% and with the money received from the sale purchased a second lot, which he sold at a loss of 5%. What was the selling price of the second lot?" Or it would solve the problem: "A merchant paid $240 for goods, marked them at 15% above cost and sold them at 5% less than the marked price. What was the selling price of the goods?"

The solution of either of these problems consists of mental addition and subtraction and written multiplications and additions. Now consider related problems in division.

In the above continued product, erase the 276 and the 240, and let it be required to find the values of the numbers erased. By dividing

262.20 by .95 we obtain 276, and by dividing 276 by 1.15 we obtain 240. In so doing we have solved the problem: "A man bought a lot for a certain amount. He sold it at a gain of 15%, and with the money received from the sale purchased a second lot, which he sold at a loss of 5%. He received $262.20 as the selling price of the second lot. What was the cost of each of the two lots?" Or we should be solving the problem: "A merchant paid a certain amount for goods. He marked them at 15% above cost and then sold them at 5% less than the marked price, thereby receiving from the sale $262.20. What was the cost of the goods and the marked price?"

```
(   )
 1.15
(   )
  .95
262.20
```

It must be understood that the above discussion has only to do with the discovery of the solution of certain inverse problems. The solution of either of these is as follows:

```
          2 76.           2 40.
(a) .95)262.20     1.15)276.0    Or
        190             230
         72 2            46 0
         66 5            46 0
          5 70
          5 70
```

(*b*) Use the cancellation form:

$$\frac{262.20}{.95 \times 1.15} = 240.$$

The subordinate problems, or steps. The methods of attack to be used in solving the subordinate problems, after the general plan of attack has been determined, are those used in solving independent problems with one condition, or step. The use of the equation, discussed on pp. 112–116, in connection with the solution of mechanical examples, applies equally well here, it being necessary only to understand the language of the problems. In many cases the equation is not necessary, for the chief operation in any one step may be determined from a knowledge of the relation between the direct and inverse operations as shown in the steps in the problems illustrated above.

Two forms of the equation. The first use of the equation in the solution of the following problem is preferable, but the second use has a place in case it is desired that the pupils get a full appreciation of the detail of the solution, especially in relation to objects.

Ex. If $\frac{2}{3}$ of a yard of cloth costs \$.50 how much will a yard cost?

SOLUTION:

(*a*) Let y (for yard) represent the cost of 1 yard.

We have $\frac{2}{3}y = \$.50$

$$y = \$.50 \div \tfrac{2}{3} = \$.50 \times \tfrac{3}{2} = \$.75.$$

(*b*) Cost of $\frac{2}{3}$ of a yard $= \$.50$

Cost of $\frac{1}{3}$ of a yard $= \frac{1}{2}$ of \$.50, or \$.25

Cost of $\frac{3}{3}$ of a yard $= 3 \times \$.25 = \$.75.$

\$.50

$\frac{2}{3}$ of a yard

Notice in (*b*) that the last two equations constitute the steps in the explanation.

Pupils are sometimes puzzled over the statement of the second equation above, in which it is said that since $\frac{2}{3}$ of a yard costs \$.50, $\frac{1}{3}$ of a yard costs $\frac{1}{2}$ of \$.50, etc. This should not be confusing if the phrase "of a yard" is repeated. Spelling out the word "third" makes matters clearer. To illustrate,

Cost of 2 thirds of a yard = \$.50

Cost of 1 third of a yard = $\frac{1}{2}$ of \$.50, or \$.25; etc.

Relation between percentage and fraction problems. An explanation of a percentage problem becomes an explanation of a common-fraction problem when use can be made of the aliquot-part relations. The mechanical work in the above problem solves the following problem: "A boy lost

$33\frac{1}{3}\%$ of his money and had \$.50 remaining. How much did he have at first?" The discussion on pp. 109–111 applies equally well to content problems in fractions and percentage.

An illustration with time. The data in problems are taken frequently from events in life. Each event must necessarily, in point of time, follow some other event. The data in such problems illustrate that multiplications correspond to an advance in time, while divisions correspond to a retrogression in time. The idea is shown in the following examples:

Ex. A farm was bought for \$20,000. It was kept for 2 years, when it was sold at an advance of 20%. The money was invested in city property, which was sacrificed 6 months later at a loss of $33\frac{1}{3}\%$. What was the selling price of the city property?

Increase of time. →

(1)	(2)	(3)
Farm bought \$20,000	Farm sold City property bought	City property sold

With reference to the diagram, we notice that the events occur in the order (1), (2), (3). The value of (1), to speak somewhat inaccurately, is known. The progress of time and the order of the events corresponding, (2) and (3) are found by continued multiplications. The first of the following solutions indicates the process and is the one to use where the per cents are not reducible to simple fractions. The second solution takes advantage of the aliquot-part relations.

SOLUTIONS:

$$(a)\quad \begin{array}{r} \$20000 \\ \underline{1.20} \\ 24000.00 \\ \underline{.66\frac{2}{3}} \\ \$16000.00 \end{array} \qquad\qquad (b)\quad \begin{array}{r} 5\underline{)\$20000} \\ \underline{4000} \\ 3\underline{)24000} \\ \underline{8000} \\ \$16000 \end{array}$$

Ex. A man sold some goods for \$54 after marking them down first 25% and later 10%. What was the original marked price?

Increase of time.

$\longrightarrow$

(1)	(2)	(3)
Original marked price	Next marked price	Final marked price, or selling price, \$54

The value of (3) is known. The events occurred in the order (1), (2), (3). Successive divisions will give the value of (1).

SOLUTIONS:

(a) $.90\overline{)54.0}$
$\quad 60$

$.75\overline{)60.00}$
$\quad 80$

(b) $9\overline{)54}$
$\quad 6$
$3\overline{)60}$
$\quad 20$
$\quad 80$

(c) $$\frac{\cancel{54}}{\cancel{.75} \times \cancel{.90}} = \frac{1}{.0125} = 80$$

(d) $.75 \times .90 = .6750$

$.675\overline{)54.000}$
$\quad 80$

(e) $$\frac{54}{\frac{3}{4} \times \frac{9}{10}} = \frac{54 \times 4 \times 10}{3 \times 9} = 80.$$

(*f*) Let x represent the original marked price.

Then $.90\,x =$ the next marked price,

and $.75$ of $.90\,x =$ the selling price.

Therefore $.75 \times .90\,x = 54$,

and $$x = \frac{54}{.75 \times .90} = 80.$$

(*g*) In (*f*) use $\frac{3}{4}$ for .75 and $\frac{9}{10}$ for .90; then

$$\tfrac{3}{4} \text{ of } \tfrac{9}{10}\,x = 54,$$

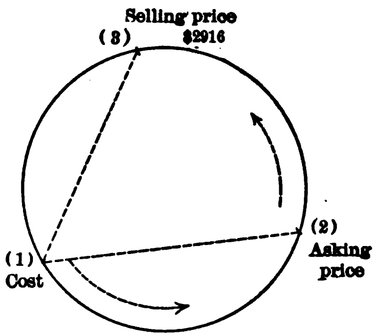
and $$x = \frac{54}{\frac{3}{4} \times \frac{9}{10}} = \frac{54 \times 4 \times 10}{3 \times 9}.$$

$$x = 80.$$

Observe that (*f*) gives the explanation for (*c*), and (*g*) for (*e*).

Another illustration. The accompanying diagram may prove suggestive for the solution of problems like the following old-time textbook problem.

A man bought goods for a certain amount. He asked 35% above the cost, but finally sold them for \$2916, which was a gain of $21\frac{1}{2}$% on the cost. What was the cost and the asking price?



SOLUTION: The order of occurrence of the events is (1), (2), (3), as in the diagram. The value of (3) is known and there is a specified relation between (1) and (3); that is, the selling price is $121\frac{1}{2}$% of the cost. The cost is found by dividing \$2916 by 1.215, giving \$2400. Since the asking price was 35% above cost, we get 135% of \$2400, or \$3240, for the asking price.

A generalization. From the above illustration it is seen that it is necessary to know three items of data in such problems in order to obtain three other items, the six items corresponding to the total number of lines and points in the diagram. The points represent the concrete numbers of the problem and the lines relations (ratios) between the numbers represented by the points. The value of at least one point must be known if it is desired to find the values of the other points. The principle may be extended by selecting, for illustration, a greater number of points and lines on a circle. The table shows the data necessary for a complete solution:

Number of Items in Data	Number of Items to be Found
3	3
4	6
5	10
n	$\frac{n(n-1)}{2}$

Choice of methods. The method to be used in the solution of a problem depends to a great extent upon the numbers involved. If the per cents are reducible to simple fractions, the work may be much abbreviated. Generally where several steps are involved it is best to use cancellation. Sometimes it is necessary to use the equation in order to be sure of a particular step, but in many cases, the process being familiar, the equation is superfluous. It must be understood that whenever the equation is used there is still need, unless the operations are simple, of multiplications and divisions on the side.

COMMON COMMERCIAL APPLICATIONS OF PERCENTAGE

Percentage and its applications. In teaching the applications of percentage, if we add to an understanding of the three types of percentage problems an understanding of business terms and usages, we shall find in the various topics the same mechanical processes plus the necessary business interpretations. Topics like profit and loss, commission, and commercial discount are very simple, both from the business and mathematical points of view, and hardly deserve to be classified as separate topics.

Profit and loss. Tradition has held profit and loss

as a separate topic in percentage. Perhaps there is a gain in having buying and selling problems thus classified, but it is scarcely consistent to give pupils such problems in the first work and then, as some texts do, again give them the same kind of problems under the title "profit and loss."

1. *Relation to previous work.* Little need be said here concerning the solution of problems under this topic, for the ground has already been covered under the preceding topic. The mechanical work in all profit and loss problems is based upon the principles found on pp. 107–116. It may be well, however, to repeat in this connection the use of the terms "base," "rate," and "percentage." In any step of a solution involving multiplication, the multiplicand is called the base; the multiplier the rate; and the product the percentage. These terms are not used so much as formerly, but they serve for purposes of description in discussing principles.

2. *A suggestion concerning method.* A matter of expediency in solving profit and loss and other percentage problems should be considered here. For the sake of ease in learning to solve inverse problems, work the direct process (multiplication) by the first of the following methods:

Ex. A man bought a house for \$3500 and sold it at a loss of 5%. What was the selling price?

SOLUTION: (*a*)
$$\begin{array}{r} \$3500 \\ \underline{.95} \\ 175 \\ \underline{315} \\ \$3325 \end{array}$$

(*b*)
$$\begin{array}{r} \$3500 \\ \underline{.05} \\ \$175 \end{array} \qquad \begin{array}{r} \$3500 \\ \underline{175} \\ \$3325 \end{array}$$

INVERSE PROBLEM: A man sold a house for \$3325, thereby losing 5% on the cost. What was the cost?

SOLUTION:

$$
\begin{array}{r@{}l}
 & \$35.00 \\
.95) & \overline{\$3325.00} \\
 & \underline{285} \\
 & 475 \\
 & \underline{475}
\end{array}
$$

In a problem of the type of the first one above, where it is required to find both the gain and the selling price, the form (*b*) is to be preferred. It is also the form that pupils will first use. Again, in an inverse problem where it is required to find only the gain or loss, it is generally advisable to find the gain or loss directly, without first finding the cost. Note the ease of the first of the following solutions:

Ex. A man sold a house for \$2200, thereby gaining 10%. What was the gain?

SOLUTION:

$$
(a)\quad \begin{array}{r@{}l} 11) & \underline{\$2200} \\ & \$200 \end{array}
\qquad
(b)\quad \begin{array}{r@{}l} 1.10) & \underline{\$2200} \\ & \$2000 \end{array}
\qquad
\begin{array}{r} \$2200 \\ \underline{2000} \\ \$200 \end{array}
$$

An explanation which gives (*a*) is:

Let *c* represent the cost. Then using common fractions we have:

$$\tfrac{11}{10}\,c = \$2200$$
$$\tfrac{1}{10}\,c = \tfrac{1}{11} \text{ of } \$2200 = \$200$$

But $\frac{1}{10}$ of the cost is the gain. Therefore $\frac{1}{11}$ of \$2200, or \$200, is the gain.

Commercial discount. Commercial discount should be taught as one of the earliest applications of percentage. The simplest phase may be introduced as early as the fifth grade, for at that time pupils are old enough to under-

stand what merchants mean when they advertise specials at 25 % or $33\frac{1}{3}$ % discount. Commercial discount in the broader sense has to do with transactions between the retailer and the wholesaler.

Change of base. But one aspect of the harder problems needs to be considered here. This relates to change of base and is of fundamental importance in all applications of percentage.

Ex. A merchant was allowed discounts of $33\frac{1}{3}$, 20, and 5 on goods listed at \$ 360. How much did the goods cost him?

Explanation: If there had been but one discount, $33\frac{1}{3}$ %, the goods would have cost the merchant $66\frac{2}{3}$ % ($\frac{2}{3}$) of \$ 360, or \$ 240. But he was allowed other discounts. The 20 % discount, by virtue of business practice, is figured on \$ 240, the new base. Hence \$ 240 is the multiplicand in the next multiplication. Finding 20 % of \$ 240 and subtracting is the same as finding 80 % of \$ 240. Then 80 % of \$ 240, or \$ 192, would have been the cost of the goods if there had been no third discount. Deducting 5 %, we get 95 % of \$ 192, or \$ 182.40 as the cost of the goods.

(*a*)	\$ 360	(*b*)	3)\$ 360.	List price
	.66⅔		120.	
	\$ 240.		5)\$ 240	1st possible S.P.
	.80		48	
	\$ 192.00		\$ 192	2d possible S.P.
	.95		.95	
	960		960	
	1728		1728	
	\$ 182.40		\$ 182.40	Selling price

Form (*a*) is to be preferred in the sense that it illustrates a mental subtraction of per cents to get $66\frac{2}{3}$ %, 80 %, and 95 %. This is better than to multiply by $33\frac{1}{3}$ %, 20 %, and 5 % with the accompanying subtractions, for the continuity of the process is not broken and the work is usually shorter. Form (*b*) follows this principle in the last step but elsewhere takes advantage of the aliquot-part relations, which this particular problem permits.

Change of base is illustrated also in the solution of the following problem:

Ex. A man received an estate worth \$80,000. He immediately sold $\frac{1}{4}$ of it and gave $\frac{1}{3}$ of the remainder to charity. After one year the value of the part still held by him had increased $12\frac{1}{2}\%$ in value. What was the value of his property at the end of the year?

Commission. The problems in commission that are practical are easy to understand, for they involve but a simple multiplication. It is a rare thing in these days to find texts giving problems like the following as practical examples in commission:

Ex. An agent received \$2520 to invest in goods, first taking out his commission of 5% for buying. What is his commission and what is the value of the goods?

Agents are usually allowed a salary plus commission, the commission being figured on the value of the goods bought or sold. The above problem is unpractical in that business houses do not send a specific amount to cover both the commission and the value of the goods. The solution of the problem is found from observing that \$2520, which includes both the value of the goods and the commission on that value, is 105% of the value of the goods. The following problem is more practical:

Ex. If the salary of a traveling salesman is \$18 a week plus a commission of 3% on sales, how much does he earn in a week in which his sales amount to \$1016.25?

Taxes. It is safe to say that with a full understanding of the simple applications of percentage, pupils who never studied taxes in school would find no difficulty later in verifying the correctness of their tax receipts. As with some other topics, we teach taxes less for the practice it affords

pupils in the use of arithmetic than for the information they get from its study.

Let the teacher, assisted by the pupils, get current tax rates and find what must be paid by a person owning property in their particular city or district. In doing this, prepare a list showing for what purposes any special taxes are levied, together with the corresponding rates and the rates for the state and county (or city).

IN RELATION TO LENDING AND BORROWING MONEY

Grouping of topics. Wherever topics are closely associated with some business activity they should be grouped with reference to this common center of interest unless the sequence of the arithmetical development prevents. Fortunately there is much latitude for this grouping in the applications of percentage, for the mechanical processes involved are already understood. Interest, partial payments, and discount (of notes) are the principal topics concerned in problems arising out of lending or borrowing money. The promissory note plays an important part in these problems and may well be taken as a center (or line) of interest. The necessity for notes is first shown when money is loaned. The question of the justice of interest next arises. The note is again studied when partial payments are made on account of money borrowed, and also when discount is paid by the person who has loaned money and secures payment before the money is due. Make the work as real as possible, having the class go through all the necessary business forms in these three stages in the life history of a note.

SIMPLE INTEREST

The justice of interest. Suppose Mr. A has the use of Mr. B's horse for a number of months. No one questions the justice of payment for the use of the horse. Suppose Mr. B sells his horse for $100 and lets Mr. A use the money. It must be evident to every pupil that Mr. A should pay Mr. B for the loan of the $100.

No "method" needed in first work. In a problem like "What is the interest on $400 @ 7% for 2 years?" no so-called "method" need be employed, especially in first work. This problem differs from previous problems in percentage in having the element of time introduced. The interest is found for one year and the result is multiplied by 2. Pupils should know at least two methods in interest. The following are recommended:

Proportion method. The method first discussed is sometimes called the method of aliquot parts, the two months' method, or the sixty days' method, but the term "proportion method," which is used here, would be better because principles of proportion are so systematically employed. All other things being unchanged, the interest on money is proportional (*a*) to the time and (*b*) to the rate.

For example, if the interest for 2 months is $10, the interest for 8 months is 4 times $10, or $40. Again, if the interest @ 6% is $15, the interest @ 2% is $\frac{1}{3}$ of $15, or $5.

In employing the proportion method, first find and tabulate the interest on the principal for 2 months at 6% per annum. Then do the same for 6 days. Since the interest on $1 for 2 months @ 6% is $.01, the interest on any principal for 2 months @ 6% is found by moving the decimal point in the principal two places to the left. Again, since 6 days is $\frac{1}{10}$ of 60 days, or 2 months, the interest for 6 days is found by moving the decimal point in the principal three places to the left.

Ex. Find the interest of \$524 for 7 mo. 14 da. at 8% per annum.
SOLUTION:

\$ 5.24	Int.	2 mo.	@ 6%	
.524	Int.	6 da.	@ 6%	
\$15.72	Int.	6 mo.	@ 6%	(3 × \$5.24)
2.62	Int.	1 mo.	@ 6%	(½ of \$5.24)
1.048	Int.	12 da.	@ 6%	(2 × \$.524)
.175	Int.	2 da.	@ 6%	(⅓ of \$.524)
\$19.563	Int.	7 mo. 14 da.	@ 6%	
6.521	Int.	7 mo. 14 da.	@ 2%	(⅓ of \$19.563)
\$26.08	Int.	7 mo. 14 da.	@ 8%.	

In practice the parentheses at the right are omitted. For quick work, the tabulation may be omitted. The method permits the use of mental arithmetic. There should be no figuring on the side. Note that addition only is involved in the written work.

Observe the proportions between the last three numbers on the left and the three per cents on the right.

The following detailed plan illustrates the method further:

a. Whatever the rate of interest may be, first find the interest for the given time at 6%.

b. As a basis for computation set down the interest at 6% for 2 months, and for 6 days.

c. Add the various amounts of interest that correspond to the subdivisions of the time. These items of interest are found as follows:

If the interest is for 10 months, multiply the interest for 2 months by 5; if for 11 months, add to the interest for 10 months the interest for 1 month (½ of the interest for 2 months); if for 15 months, first find the interest for 14 months.

When the time is more than 1 year, the interest at 6% for 1 year may be written down as many times as there are years.

If the time is 12 days, multiply the interest for 6 days by 2.

If the time is 18 days, multiply the interest for 6 days by 3.

If the time is 3 days, get ½ of the interest for 6 days.

If the time is 9 days, add the interest for 3 days to the interest for 6 days.

If the time is 5, 10, 15, or 20 days, get $\frac{1}{12}$, ⅙, ¼, or ⅓, respectively, of the interest for 2 months (60 days).

d. The interest on the principal being found at 6%, find the interest at the required rate. Proceed as follows:

If the rate is 2%, get ⅓ of the interest at 6%.

If 3%, get ½ of the interest at 6%.

If 8%, add the interest at 2% to the interest at 6%.

If 9%, add the interest at 3% to the interest at 6%.

If 11%, add the interest at 3% and the interest at 2% to the interest at 6%.

The following solution illustrates how little work need be put down in using the proportion method. Instead of tabulating the interest for 2 months, look at the principal ($325) in the statement of the problem and imagine the decimal point moved two places to the left. In finding the interest for 5 days, get $\frac{1}{12}$ the interest for 2 months (60 days). In finding the interest at ½ of 1% get $\frac{1}{12}$ the interest at 6%:

Ex. Find the interest on $325 for 11 mo. 5 da. @ 9½%.

SOLUTION:

$$
\begin{array}{r}
\$16.25 \\
1.625 \\
\underline{.27} \\
18.145 \\
9.072 \\
\underline{1.512} \\
\$28.73
\end{array}
$$

Cancellation method. The cancellation method is more easily understood than the above, but it is longer and adds less to the arithmetical attainments of the pupil.

Ex. Find the interest of $524 for 7 mo. 14 da. @ 8%.

SOLUTION: $$\frac{224 \times \cancel{8} \times \$524}{\underset{45}{\cancel{360}} \times 100} = \$26.08$$

In writing down the values on the line, first write $524. Multiply by $\frac{8}{100}$ to get the interest for one year, or 360 days. Next divide by 360 to get the interest for 1 day. Then multiply by 224 to get the interest for 224 days (7 mo. 14 da.).

SIDE WORK:

$$
\begin{array}{r}
524 \\
\underline{224} \\
2096 \\
1048 \\
1048 \\
4500)\overline{117376.}(26.08 \\
273 \\
376 \\
16
\end{array}
$$

The cancellation method is especially good in case exact days are counted and the year is taken as 365 days.

If the reader cares to count the number of figures in each of the above solutions, he will find the proportion method exactly 45% shorter than the cancellation method. The tabulations in the first method are of course not counted, excepting the interest values for 2 months and for 6 days, and these are not absolutely essential in the solution itself. In the second solution a short method of dividing was used and the quotient was written at the side, instead of above, to avoid the necessity of repeating the product. Where considerable figuring in interest is to be done, it is clear that a short method is a time saver.

Other methods. The so-called six per cent method is long and cumbersome and, on account of the small fractions that arise, is open to inaccuracy. Compare with the first method above in the latter regard. The second method following is better than the six per cent method:

Ex. Find the interest of \$524 for 7 mo. 14 da. @ 8%.

(*a*) Six per cent method.

SOLUTION:

.06	Int. on \$1, 1 yr. @ 6%	
.005	Int. on \$1, 1 mo.	
$.000\frac{1}{6}$	Int. on \$1, 1 da.	
.035	Int. on \$1, 7 mo.	\$524
$.002\frac{1}{3}$	Int. on \$1, 14 da.	$.049\frac{7}{9}$
$6).037\frac{1}{3}$	Int. on \$1, 7 mo. 14 da. @ 6%	407
$.006\frac{2}{9}$	Int. on \$1, 7 mo. 14 da. @ 1%	4716
8		2096
$.049\frac{7}{9}$	Int. on \$1 for 7 mo. 14 da. @ 8%	\$26.08

(*b*) Another method.

SOLUTION:

\$524		\$3.493	
.08		$7\frac{7}{15}$	
12)41.92	Int. for 1 yr.	1630	
\$3.493	Int. for 1 mo.	24451	
		\$26.08	Int. for 7 mo. 14 da. ($7\frac{7}{15}$ mo.).

This method is good except when the multiplier contains an unwieldy fraction.

Use of exact days. Sometimes notes specify that the principal is due a number of days from date. In such cases the exact number of days must be found between dates, taking 365 as the number of days in a year. In finding the difference in days, the custom is to count the number of days remaining in the month in which the note was drawn and add in turn the number of days in the months following, up to and including the day when the interest ceases. In solving problems in which the exact number of days is used, employ the cancellation method.

Compound interest. 1. *Relation to simple interest.* Perhaps more notes bear compound interest than simple interest, yet the need of computing compound interest is relatively small because the interest on notes is commonly paid when due. Hence in most cases the problem in arithmetic is merely one of simple interest every time the interest is paid, the principal always being the face of the note. In case interest is not paid when due, there arises a chain of problems in simple interest when the note is to be paid off, interest being figured on constantly increasing principals.

2. *The period of compounding.* When the interest on a note is compounded annually for a whole number of years, the interest remaining unpaid, there are as many problems in simple interest as there are years. If there are also months and days, there is one additional problem. In the problem, "Find the interest on $240 for 3 yr. 8 mo. 10 da. at 10% per annum, interest compounded annually," there are four problems in simple interest.

Interest is frequently compounded semiannually and quarterly. When a note is compounded quarterly for a whole number of years, the interest remaining unpaid, there are as many problems in simple interest as there are quarters of a year in the number of years the note has run. If there are also months and days (less than 3 months), there is one additional problem. In the problem, "Find the interest on $860 for 1 yr. 7 mo. 9 da. at 9% per annum, interest compounded quarterly," there is a chain of seven problems in simple interest, the last having 1 mo. 9 da. as the time element. In the first six problems the rate is $2\frac{1}{4}$%, for 9% a year is equivalent to $2\frac{1}{4}$% for 3 mo.

It is advisable to use ordinary multiplications, as in the following solution of the above problem, excepting for the last step.

SOLUTION:

$860	The amount due at the end of 1 yr. 6 mo. is $982.83. Now		
$1.02\frac{1}{4}$	find the interest on $982.83 for 1 mo. 9 da. @ 9% per annum.		
2 15	$4.914	1 mo.	
17 20	.983	6 da.	
860	.491	3 da.	
879.35	6.388		@ 6%
$1.02\frac{1}{4}$	3.194		@ 3%
etc.	9.58		@ 9%
	982.83		
	992.41		
	860.		
	$132.41	Compound interest.	

It is not necessary to retain more than three decimal places in the above products in order to insure an answer correct to two places.

Use of tables. Pupils should be taught how to use interest tables after they have learned to work simple and com-

pound interest without tables. The use of tables provides the same useful principles of proportion as illustrated in the proportion method of computing simple interest given above.

Inverse problems. Generally speaking, inverse interest problems are of rare occurrence in business. It is seldom necessary to find the amount of money put at interest for a given time that will yield a specified interest. Other elements being given, it is equally rare to need to find the time. But the question of what rate of interest, or income, a certain investment will yield, is of importance. In such problems the element of time may or may not enter. Pupils should have enough drill in solving inverse interest problems to meet ordinary business demands and no more.

The following equations show these related problems in simple interest:

$I = t \times r \times P$

$A = t \times R \times P$, where I = Interest
A = Amount
t = Time, usually in terms of years
r = Rate
$R = 1 + r$
P = Principal.

Compare these with the equations on p. 112 and observe that the factor t is the only essential difference.

The equation for compound interest is $A = R^t \times P$, where t represents a whole number of years and R, P, and A are used as above.

Annual interest. Annual interest is not commonly used in business and hence need not be taught in school. It is computed as follows:

(1) Find the simple interest on the principal to the time of settlement. (2) Figuring from the end of each year, get the simple interest on each of the equal yearly interests up to the time of settlement. (3) Add the results in (1) and (2). Annual interest, unlike simple interest,

requires that interest be paid annually. That compound interest is not the same as annual interest is seen from the following formulas:

$$\text{Compound Int.} = [(1 + r)^t - 1] \times P$$

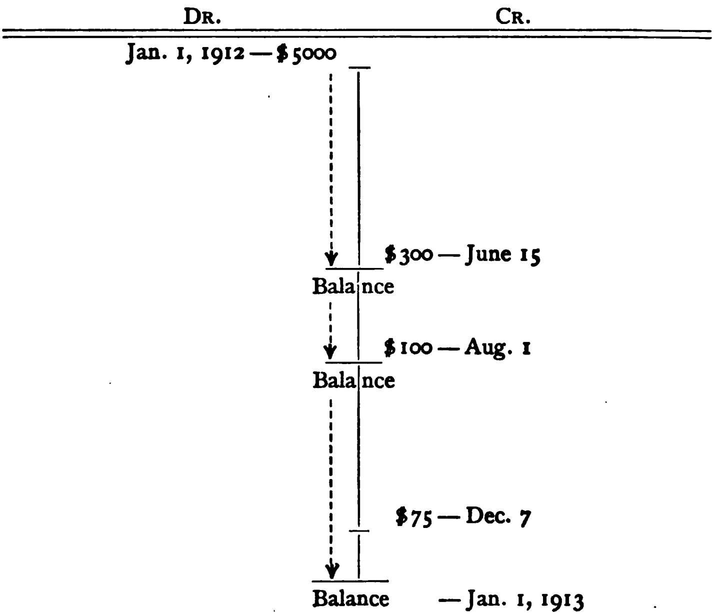
$$\text{Annual Int.} = \left[tr + \frac{t(t-1)}{2} r^2\right] \times P$$

$$\text{Simple Int.} = tr \times P.$$

PARTIAL PAYMENTS

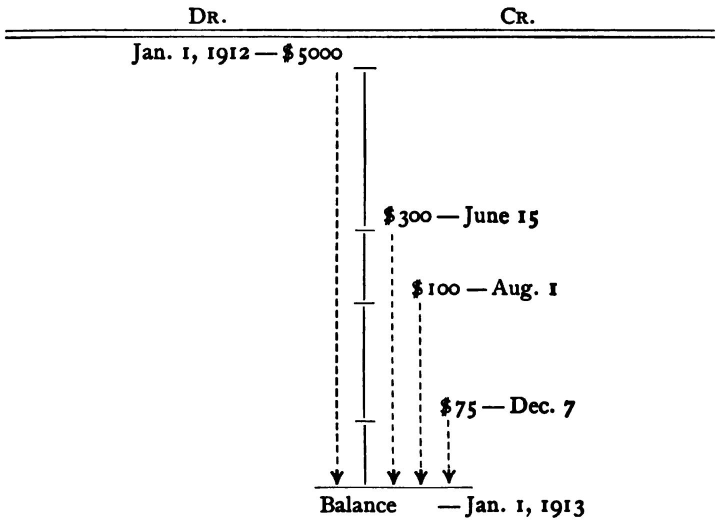
United States rule. As an aid in presenting the plan of computation in partial payments, the teacher may find it helpful to arrange the data diagrammatically.

Ex. A note for \$5000 was drawn Jan. 1, 1912, interest at 6% per annum. The following payments were made: June 15, \$300; Aug. 1, \$100; Dec. 7, \$75. How much was due Jan. 1, 1913?



A balance was not made on Dec. 7, for the payment of $75 was less than the interest due from Aug. 1 to Dec. 7. This provision is made in the United States rule to secure the borrower from paying compound interest when the note is drawn at simple interest.

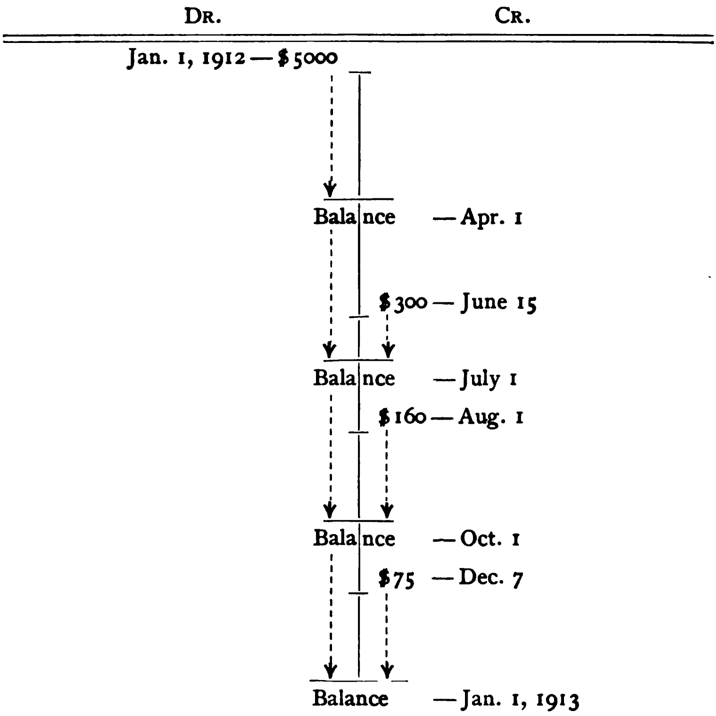
Mercantile rule. The mercantile rule is used only for short-time notes. For long-time notes, its use works an injustice to the lender. Any sum borrowed at 6 % interest may be paid off in about 25 years, if the yearly payments of interest are considered as partial payments and the mercantile rule is employed. A diagram corresponding to one given for the United States rule follows, the same problem being used:



Where compound interest is used. In case notes are drawn bearing compound interest, and partial payments

are made, the method indicated in the following diagram, which is used by some business houses, may be employed.

Ex. A note for $5000 was drawn Jan. 1, 1912, with interest at 6% per annum, compounded quarterly. Payments: June 15, $300; Aug. 1, $100; Dec. 7, $75. Find the amount due Jan. 1, 1913.



DISCOUNTING NOTES

Business practice. Notes are commonly discounted by the bank discount method, but a variety of methods are in use. The following illustrations show what is done in business practice.

1. A common way of borrowing money of a bank is as follows: In case Mr. B wants to borrow $1000 of the bank where he keeps his account, he signs a non-interest-bearing note, properly secured, in favor of the bank, due, say, in 1 year. The bank immediately discounts the note, taking in advance the year's interest, figured, say, at 6%. Mr. B pays the principal at the end of the year. This is a simple phase of bank discount.

2. A loans B $1000 for 1 year, with interest at 6% per annum. After 3 months A wants his money, so he goes to his bank to realize what he can on B's note. The bank gives A the full $1000, taking in exchange the note properly indorsed. Has A gained or lost? What does the bank gain? In what sense is the note discounted?

In answering the last question, recall that the one who takes over the note usually charges for his services.

3. A bank may accommodate one of its patrons by taking over a note, paying for it the principal plus the interest due. In this case the bank gains merely the interest for the time the note has yet to run, charging no discount.

4. Work the second example under the condition that the discount is figured on the face of the interest-bearing note for the remaining 9 months at the rate of 6% per annum. This illustrates "shaving" a note. How much does A receive as proceeds?

5. Finally work the second example by the bank discount method by discounting on the future value of the note for the remaining 9 months, at 6% discount.

Compare the results in the transactions described on

p. 151, not neglecting to consider what the bank realizes in all after B settles his indebtedness.

6. Not all banks do a discounting business. Neither is discounting confined to banks. Notes are frequently bought up without respect to the time they are to run, the rate of discount being a certain per cent of the face. For example, an insurance agent, who has written a number of policies in a certain community and has taken as first payments of premium the promissory notes of those insured, may take the notes to a discounter, who buys them and figures his discount at a certain per cent of the total face value of all the notes, irrespective of the time they are to run separately. In such cases, of course, the notes run but a few months and they are for comparatively small amounts.

A teacher cannot well drill on all the variations mentioned above, but she should keep in touch with common business practice.

Bank discount illogical. Bank discount, although sanctioned by usage, is illogical from the standpoint of the principles of simple interest. For example, in (1) above, to get the discount we figure interest on $1000, the amount to be paid the bank in the future, the bank getting its interest in advance. The borrower is out one year's interest on $60 since he pays the $60 interest one year before it is due.

IN RELATION TO THE TRANSFER OF MONEY

Kinds of money. From the informational point of view, perhaps there is nothing in the elementary school curriculum that is more important than the subject of money.

Pupils should have fairly well defined ideas of the necessity of such a medium of exchange. They should know the accepted use of the terms "money," "coin," and "currency." They should know the common substitutes for money. Most pupils are familiar with coin money, but few can distinguish between the different kinds of paper money, this being especially the case west of the Rocky Mountains, where gold and silver are the more common forms of money. Let the class see and examine treasury notes, gold and silver certificates, and national bank notes and let them gain some idea of the cause of their existence. Lead them to understand that the money order, the check, and the draft are substitutes for money that have a limited circulation.

Transfer of money. Local indebtedness is paid either by money or by check, while indebtedness elsewhere is paid by means of drafts (and occasionally checks), postal and express money orders, telegrams, telephone messages, and (for small amounts) stamps. Have the class understand the method of and reason for sending money by any of these means and discuss their relative merits. Explain the use of letters of credit.

Exchange. In a broad sense, the topic of exchange includes the features just described, but the name "exchange" is technically associated with sending money by means of drafts. There is absolutely no arithmetic necessary for an understanding of the subject of exchange as it relates to the great majority of individuals. A person who sends money by draft is told the charges and that is all there is to the matter. It is well, however, to have the class know through what hands a draft may pass before it

is finally returned to the bank that issues it. The "life" of the draft and the check may be illustrated by the pupils in imitation of what happens in actual practice, just as "business" in connection with notes may be transacted by them. The following problems, illustrative of what was in our schoolbooks not so long ago, are of no practical value to the pupil and should not be taught. It is true, however, that school boards very frequently require prospective teachers to solve such problems in examinations.

Ex. Mr. A, desiring to remit $450, bought a sixty-day draft, paying $\frac{1}{4}$ of 1% premium. How much did it cost him, money being worth 6%?

Or worse yet:

Ex. Find the face of a thirty-day draft that cost $4985, exchange being at a discount of $\frac{1}{2}$ of 1% and money worth 6%.

A school bank. Throughout the upper elementary grades have the classes experience as much of the work as possible. Have imitation business where real business is not possible. Pupils should be able to write common business paper, and even if they can do this only in imitation of business, they are learning something of value. Have the class transact the business of depositing money in and drawing money out of a bank and of sending money by means of money order and draft. All this leads to a closer knowledge of the bank. A school bank can be operated in any school. One could not find a better center of business interest than this.

In order to create business for the bank, pupils may represent various business activities and could on two or three days a week devote an hour to trade. In lieu of real money, school currency could be used. In work of

this character teachers must necessarily begin in a simple fashion. Once started, it is possible, however, to stimulate much interest among the pupils. In connection with such work there is opportunity for borrowing and lending money, giving and taking notes, making partial payments on notes, and keeping cash and possibly ledger accounts, besides learning how to do business with the bank through the use of checks, drafts, and pass books. The teacher and one or two of the best business pupils should be the officers of the bank. It is possible to have actual accounts kept in the school bank. Most schools support one or more organizations in which the pupils participate. These could deposit money in the bank. Teachers could do the same in order to encourage business and pupils likewise could open accounts and pay school organization taxes by means of checks on the school bank. Most teachers find it difficult to undertake work like this on account of the pressure of other work that they feel must be done, but if only a small part of the above suggestion is carried out, the interest aroused in the pupils and their reaction towards the related work in arithmetic will prove the venture worth while.

IN RELATION TO INVESTMENTS

Value of problems of comparison. There is opportunity in the study of arithmetic for pupils to get some usable knowledge concerning investments. The textbook topics that are especially concerned are stocks, bonds, and insurance. Unclassified problems concerning the same general topic should also be assigned. The following question may arise: Is it cheaper to build or to rent? Get local

data and assign a problem to correspond. One of the first things a pupil learns in connection with such a problem is that when money is invested in a house, besides figuring the cost of maintenance, insurance, and taxes as balancing rental, the loss of interest on money put into the house must also be considered.

Many of the problems in arithmetic may be improved by thus bringing in the element of comparison. We may ask:

Is it better to loan $1500 at 8% for 9 months at simple interest or to loan it at 7% for the same time at compound interest, interest payable quarterly?

Again:

Is it better to rent a horse for a camping trip by paying $.75 a day for 30 days or to buy the horse for $80, figuring on selling him afterwards for $70, after boarding him for 10 days extra at $1 a day in order to get him in condition?

Stocks. Teachers are agreed that stocks and bonds should not be taught from the standpoint of the exchanges. The average citizen does not need to understand about margins nor how brokerage is figured. But there are some simple applications of arithmetic and especially some matters of information that should be considered in the school.

Most communities have their local stock companies. It is a common occurrence in the life of almost any citizen to be asked to buy stock in some concern. There may be a local company that is seeking to develop a mine or promote an oil project, or, in the larger cities, it may be that some manufacturing plant is awaiting development. Again, the mails are flooded with letters offering stock for sale in numerous safe and unsafe enterprises.

The school should give the prospective citizen some idea of what it means to own stock in a company, and how the income is figured. Small local stock companies are frequently organized under the following conditions: Mr. E discovers and partially develops a mine. Not having sufficient funds, he gets Mr. H to help develop the property and erect a mill, giving him in payment one half interest in the property. Further development work being required and a larger mill being necessary, Mr. E and Mr. H decide to organize a stock company to help finance the concern. They decide to issue $50,000 worth of stock to represent the value of the property. They take $20,000 worth of stock each, and offer for sale the remaining $10,000 worth, which is commonly called treasury stock. If they need only $5000 in money, they may sell this stock for $.50 on the $1. Stock in large companies is usually divided into shares of the par value of $100 each, but in small companies, like the above, shares may be $10 or $1 each in order to insure more ready sales among the small investors.

Suppose Mr. C buys 50 shares of the above stock at $5 a share, the par value being $10. In a year's time the company prospers and declares an 8% dividend. This means that 8% of $50,000 (par value) is distributed among the stockholders. Mr. C has $500 worth of stock, and hence receives 8% of $500, or $40, as his dividend. In case a 10% assessment had been levied, Mr. C would have paid 10% of $500, or $50, to the company. In case Mr. C sells his stock, say for $7 a share, he will gain $2 a share, or $100 on his 50 shares, making no allowance for dividends or assessments.

Watered stock. Pupils can readily understand the practical significance of the arithmetic involved in the above, but they will not appreciate any further details of dealings in stocks. The teacher should be prepared, however, to tell any inquisitive pupil about bulls and bears, common and preferred stock, and the watering of stock. A company is said to water stock when it increases its capital stock to an amount far in excess of a fair value in order to keep the rate of dividend below a certain minimum per cent, to comply with the laws of some states.

Bonds. The average citizen has more need to understand about bonds than about stocks for the reason that while only a limited number of people invest in stock companies, every voter is required at some time to vote on improvement bonds and every taxpayer must stand ready to pay his or her share of the increase of taxes caused by the issuance of bonds.

A bond is but a formal promissory note. Notes are made out usually at the time money is borrowed. Bonds are issued before the money is borrowed. They state the rate of interest that will be paid and the number of years the separate bonds are to run. The bonds are then advertised for sale. The highest bidder is the one who agrees to pay the borrower the greatest amount of money above the face of the bonds. This bonus, or premium, is not repaid by the borrower.

A stock company may raise money in two ways. One way is from the sale of stock, where the buyer of stock becomes one of the stock company. The other way is from the borrowing of money, the company giving its

notes as security, or, as it is commonly expressed, selling its bonds. The buyer of bonds becomes a lender of money to the company. It is common for stockholders also to be bondholders of the same company.

District school bonds, city improvement bonds, and United States government bonds are bonds that the public is interested in. The pupils should see and examine such bonds. The town treasurer or possibly the clerk of the school board can furnish some bonds that have been paid off, or redeemed, or perhaps some friend can supply a United States bond (coupon preferred) that is still "alive." The following questions may be asked with reference to a coupon bond under examination: When was the bond issued and what is the last date on which it may be redeemed? For what specific purpose was it issued? What is the par value of the bond, the rate of interest, and the number of times a year that the interest is payable? What is the amount of interest written on each coupon and the date when the interest on each is payable? How many coupons have been cut from the bond? See if the answers to the last two questions agree with the solution of the problem that can be made after examining, for example, the data in a United States Spanish War bond of 1898.

In both stocks and bonds, dividends and assessments are figured on par value and not on market value. The problem discussed above under stocks (which was drawn from actual experience) illustrates this point. For example, if a man buys 6% stock at $5 ($10 being par value), he is making $.60 a share or 12% interest on money invested.

CHAPTER VI

THE APPLICATION SIDE OF ARITHMETIC — FORMS AND MEASUREMENTS

DENOMINATE NUMBERS

Use of the term. By denominate numbers we mean, strictly speaking, concrete numbers, but the textbooks use the term with respect to those groups of concrete numbers with which tables are associated. We speak of tables of denominate numbers. In 5 yd. 2 ft. 9. in., we have a compound denominate number, while 5 yd., 2 ft., and 9 in. are individually denominate, or concrete, numbers. In the texts of twenty years ago 5 yd. 2 ft. 9 in. was called a compound number. It was well named, for it consists of three different kinds of concrete numbers in which the regularity of the ratio between the units as in ordinary numbers (10 to 1) is lacking.

Distribution of the work. In former years the tables of denominate numbers, with the exception of the simplest, were taught as a topic by itself. To-day the tables and their applications are distributed throughout the grades. Some texts go to the extent of omitting the term "denominate number" altogether, a circumstance which does no harm as long as the necessary tables are taught and applied.

Value of the tables. In another connection (pp. 246–248) we discuss the question of eliminating obsolete and

waste material from the course in arithmetic and refer to a number of tables that could be (and sometimes are) omitted with profit. Perhaps more time has been wasted in teaching and applying useless tables than in any other part of arithmetic.

Teaching the tables. The method of procedure in teaching the tables is (1) development, (2) memorizing, and (3) application. Memorizing a table without having first constructed it often leads to meaningless applications, because the pupils have no sense experience upon which to base their judgment. It is impossible, of course, to have the class construct all the tables in full, and it may sometimes be inadvisable. For instance, they may take it for granted that there are 5280 ft. in a mile, but they can easily build up the first part of the table of linear measure, 12 in. = 1 ft., 3 ft. = 1 yd., etc. The relation between the inch, foot, and yard is brought out in the early work through measurement, as is also the relation between the pint, quart, and gallon. It is imperative that the pupils have common measures like these in the schoolroom and use them.

We shall briefly illustrate the building up of the table of surface measure. The class should already know the table of linear measure and they should know the meaning of square inch, square foot, and square yard and be able to make a drawing of each. The first problem given the class is to find the number of square inches in a square foot. Under the direction of the teacher the pupils may take turns in making the drawings on the board. Let them draw a line 12 in. in length; build a square having

this line for a side; divide the square into square inches. How many square inches are there in any one of the horizontal rows? How many rows are there? Then how many square inches are there in a square foot? *Ans.* 12 × 12 sq. in., or 144 sq. in. In a similar fashion draw a square yard and have the class find the number of square feet into which it may be divided. The table may be written on the board as fast as the relations are discovered. The pupils may discover before this that parts of the table of surface measure may be built from the table of linear measure by squaring the numbers in the latter table. This is made evident when the two tables are placed side by side. Thus,

144 sq. in. = 1 sq. ft.	12 in. = 1 ft.
9 sq. ft. = 1 sq. yd.	3 ft. = 1 yd.

Care should be taken that the pupils do not say that 12 in. × 12 in. = 144 sq. in.

Applying the tables. The textbooks of to-day are for the most part applying the tables in a rational way. Most of the work formerly given in the addition, subtraction, multiplication, and division of compound denominate numbers is of little practical value. One of the few cases of practical value arises in finding the difference between dates, which is needed in computing interest. Another instance is in longitude and time, where it is necessary to multiply and divide compound denominate numbers by 15 (or 4), but even here there is some question as to the extent to which this work is of value.

A question of principle. A matter of importance to the

teacher may here be mentioned in connection with the addition, subtraction, multiplication, and division of compound denominate numbers. The same principles hold as in the same operations with whole numbers and decimals, but with the variation that in compound numbers there is a relation between units other than the 10 to 1 ratio that holds between the units of the consecutive orders in our decimal system.

$$3\overline{)8\text{ ft. }3\text{ in.}}$$
$$2\text{ ft. }9\text{ in.}$$

Let it be required to find $\frac{1}{3}$ of 8 ft. 3 in. When 8 is divided by 3, the remainder 2 must be multiplied by 12, the ratio between the foot unit and the inch unit, and 3 must be added to the product before we can again divide by 3. Compare this with ordinary division in the example, 834 ÷ 3. When 8 is divided by 3, the remainder 2 must be multiplied by 10, the ratio between the hundreds' unit and the tens' unit, and 3 must be added to the product before we can again divide by 3.

$$3\overline{)834}$$
$$278$$

The metric system. It cannot be expected that the metric system of weights and measures will be taught to any extent in schools unless it becomes of common use in business. The United States government has legalized the system, but has not provided for the enforcement of its use. It is not in conformity with sound educational doctrine to compel a pupil to study a subject that has no common usage or that does not serve as a foundation for something that will add to and complete his life. Therefore it must be a waste of time to teach the metric system in the schools with the hope that some day the public will adopt its use. Opportunities may arise, however, for use of some of the metric units in schools that attempt simple experiments in science, more particularly in connection

with the physical sciences. It seems wise at the present time to restrict the study of the metric system to the laboratory. Whenever it is taught, it should be as a system within itself. It is profitless to have pupils work examples back and forth from the metric system to our present system. There is an analogy in the study of the languages. We acquire a foreign language with the most profit to ourselves when we learn to think in that language and do not, as is too often the case, translate it into the mother tongue in order to get the thought.

BUSINESS FORMS AND ACCOUNTS

The business paper of daily life. The average citizen, although he may not be in business, has occasion to know, among other things, about bills, receipts, money orders, checks, drafts, and promissory notes. This being true, it is the duty of the school to make pupils familiar with the forms and uses of these kinds of business paper.

Its place in the school. The plan of giving a period of the day to the writing of business forms has little to commend its use. Such work should be done, wherever possible, in connection with the related topics in arithmetic or with bookkeeping. Notes should be considered in connection with interest, partial payments, and bank discount; while money orders, checks, and drafts should be studied in connection with the payments of indebtedness. The use of receipts may be brought in with either of the above lines of work or in connection with bills. The writing of all business paper should, wherever possible, be in connection with actual or fictitious school business.

Need of emphasizing important features of business paper. Some features of the reading of business paper should be made clear to the pupils. Show them, for instance, the need of stating in a note when the principal is to be paid. In bills, point out the part played in the business transaction by the persons or firms whose names appear on the headings. Who is owed, and who is owing? In like manner explain the functions of the persons or firms whose names appear in drafts, checks, and notes.

Accounts. 1. *The tendency.* During the last ten or fifteen years there seems to be a reaction in the schools with respect to bookkeeping. In former years both single and double-entry bookkeeping were required in the elementary school. Too much work was required, and it was done in a formal and perfunctory manner. Teachers and boards of education have, for the most part, concluded that such bookkeeping has little value for pupils of any grade. The other extreme has been reached in some sections of the country, for it is not uncommon to find counties and cities where pupils are not taught even how to keep a cash account.

2. *Need of bookkeeping.* There is a sensible mean between these two extremes. Single or double-entry bookkeeping taught formally from a text gives no pupil, either of the elementary or high school, any lasting idea of business practice. At the same time children should not be allowed to pass through the grades without knowing how to keep their personal cash accounts. The study of accounts in the elementary school should do at least so much for the pupil. It would also seem wise to let the

boy of the rural school learn to keep the accounts of his father's farm. Scientific farming is not scientific without strict attention to the keeping of the accounts of the farm. A farmer should know what profit a certain field is giving him or whether he has a good investment in a particular herd of cattle. The time is probably coming when the rural communities will demand that their boys learn at school enough of accounting to be of service in running their farms. With the new interest in home economics, there is a need of understanding the keeping of home accounts, and the school should also take this into consideration.

LONGITUDE AND TIME

Value of the study. The chief aim in studying longitude and time is not to be able to get differences of longitude and of time and to change differences of longitude to differences of time, and *vice versa*, although some of this work is necessary. It is more important that pupils learn the effect of the revolution of the earth on the time of day. In particular it is important that they learn how to regulate their watches in traveling from east to west and from west to east and why they do so; that they learn when to expect election returns from the east or west and why; that they learn of the need of an international date line; and that they know the meaning of standard time and can bound the four time belts in the United States. It matters little whether longitude and time be taught as a topic in arithmetic or as a part of geography. In any case the subject should be approached from the standpoint of astronomical geography, as is shown in the discussion that follows.

Steps in the development. The leading steps necessary for an understanding of the practical features of longitude and time are as follows:

1. The earth revolves from west to east.
2. A revolution through 360° produces a change of 24 hours in time. Therefore a revolution of 15° produces a change of 1 hour of time.
3. Places farther east see the sun first.
4. Hence places farther east have later time (in the day.)

The above steps give the basis for arithmetical drills in finding differences in longitude and in time, and in changing differences in longitude to differences in time, and *vice versa.*

Later and earlier time. Some matters give pupils difficulty in the study of longitude and time, particularly the question of later and earlier time. A globe placed in the proper astronomical position may be used for illustrative purposes. The experiences of the pupils may be utilized, as in the following questions:

What was the time at our sunrise this morning? (Perhaps 6 A.M.)

In what direction does the earth revolve on its axis?

In what direction from us is a place now having sunrise? (West, because the earth revolves from west to east.)

What time has that place now? (6 A.M., the same time the sun rose for us.)

What time have we now? (Perhaps 9 A.M.)

Which of the two places has the later time in the day at this instant?

Which place is the farther east?

Have places east of us later or earlier time in the day than we have? (Later, for they see the sun rise first.)

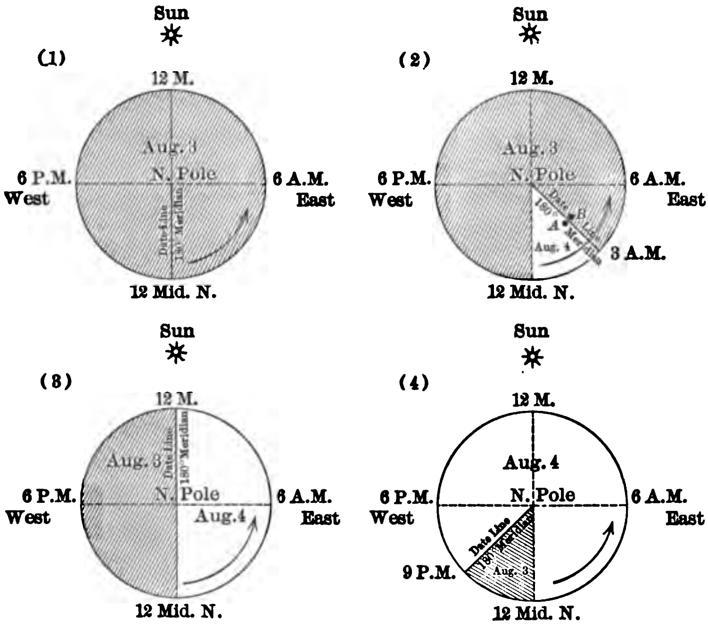
In general then, which of two places having different longitudes has the later time? the earlier time?

Beginning of the new day. The international date line. A matter not needed in ordinary computation, but one which is interesting from the informational point of view, is that of the need of a date line on the earth's surface for determining the places on the globe that are the first to begin the new day. One common point of agreement is that the new day begins for each place the world over at midnight. Thus the instant the 3d of August ends, at midnight, the 4th of August begins. Since places having different longitudes have their midnights at different times, what places having the same longitude are the first to have the 4th of August as the new day? By common agreement of the civilized nations, a line approximating the meridian of 180° has been chosen as an international date line. It passes through the Bering Strait and curves south through the Pacific Ocean in an irregular manner so as to give the possessions of any one nation, as far as possible, the same day of the month.

The best way to explain the use of the date line in determining the new day is to employ the globe. The sun may be imagined fixed above the globe on the meridian in the south and the globe turning from west to east. The following diagrams, which may be used for illustrative purposes, show the use that may be made of the globe.

NOTE. — The reader must imagine that he is looking down upon the earth from the north on a line coinciding with the axis of the earth. For convenience the circle may represent the equator. The earth is supposed to revolve in the direction of the arrow. The sun remains fixed in the same plane as the equator. All places on the earth represented by the upper half of each circle have sunlight, all places in the lower half, are without sunlight. The shaded portion of each circle

represents that portion of the earth that has Aug. 3, the unshaded portion, Aug. 4. Diagram (1) shows the ending of Aug. 3 and the beginning of Aug. 4 for all places on the date line, these places then having their midnight. Diagram (2) shows that the earth has rotated 45°. All places that have had their midnight since the date line had its midnight have the new day, while all places east of the date line that have not yet had their midnight still have the old day, Aug. 3. Diagrams (3) and (4) show the diminishing number of places that have Aug. 3 while the time of day for the date line is gradually approaching midnight.



Crossing the date line and changing the day of the month. By examining Diagram (2) it may be seen that a person who travels across the date line from A to B changes his date from Aug. 4 to Aug. 3 and *vice versa* when he

travels from B to A. Hence he goes back a day in his calendar when crossing the date line from west to east and skips a day when crossing it from east to west. In traveling from west to east he gains time (advances his watch) hour by hour until he crosses the date line, when he loses 24 hours at once; but in traveling from east to west he loses time (sets his watch back) hour by hour until he crosses the date line, when he gains 24 hours at once.

A story is related of an Irishman who was sailing from San Francisco to Japan. He was told late in the evening of March 16 that the vessel would soon cross the date line and that, according to the practice of ships sailing across the date line from east to west, the calendar would be advanced one whole day and hence the 17th of March would be dropped out. This information so distressed the Irishman that he went to the captain and persuaded him to defer the advancing of the ship's calendar until 24 hours had elapsed. This anecdote illustrates the seeming paradox that while a day is gained (the date advanced) a day is lost (dropped out), a most natural circumstance when one remembers that human conventions have no effect on the progress of time.

Errors in form. Since to a difference of 15° of longitude there corresponds a difference of 1 hour of time, the number of hours' difference in time between two places whose difference in longitude is 60° is equal to the number of times 15° is contained in 60°, or 4. Therefore the difference in time is 4 hours. The quotient is not 4 hours but 4, just as $\$8 \div \$2 = 4$, not $\$4$.

Avoid forms like the first three of the following:

$$\begin{array}{ll}
(1)\ \ 15^\circ)\underline{\,60^\circ\ \ 45'\ \ 30''} & \qquad (2)\ \ 15)\underline{\,60^\circ\ \ 45'\ \ 30''} \\
\qquad\quad 4\text{ h. }3\text{ m. }2\text{ s.} & \qquad\qquad\quad 4\text{ h. }3\text{ m. }2\text{ s.} \\
(3)\ \ 15)\underline{\,60^\circ\ \ 45'\ \ 30''} & \qquad (4)\ \ 15)\underline{\,60\ \ \ 45\ \ \ 30} \\
\qquad\quad 4 - 3 - 2 & \qquad\qquad\quad 4 - 3 - 2 \\
\qquad\quad 4\text{ h. }3\text{ m. }2\text{ s. } \textit{Ans.} & \qquad\qquad\quad 4\text{ h. }3\text{ m. }2\text{ s. } \textit{Ans.}
\end{array}$$

The logic in the third form is not strictly correct, for an explanation demands that 60° be divided by 15°, 45′ by 15′, and 30″ by 15″. The fourth form has brevity and no inconsistencies of form.

MENSURATION

First work. Mensuration is primarily concerned with finding lengths of lines, areas of surfaces, and volumes of solids. It is, in a broader sense, the computation side of geometry.

It has its basis in the measuring exercises of the early years when children work with the common units of length and learn the properties of the square and the rectangle. Some of the work in mensuration is taught under denominate numbers, for example, in developing and applying the tables of linear, surface, and solid measure.

Method of development. Most of the formulas in mensuration should be developed experimentally by using objects or drawings and following the inductive method of approach. Proper aims given the pupils by calling their attention to some practical problems within their experience add much to the success of the work. A boy in a rural district may be interested in learning how to find the areas of fields of different shapes or in knowing how to determine the number of gallons of water in his father's tank.

Areas of plane figures. 1. *The rectangle.* In the mensuration of plane figures begin with the rectangle. The inductive method will be illustrated in the following development of this simple formula. In connection with the other formulas to be considered here, the treatment for the most part will be in the nature of suggestions.

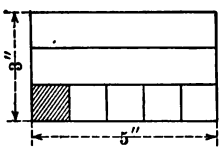
The teacher may provide an unmarked rectangular board or cardboard with dimensions 3 in. by 5 in. The object is better than a drawing on the board, for it can be handled and examined by the class. The formula may be developed through questioning, as follows:

How many square inches are contained in this rectangle? (This direct question leads the class to think for themselves. If there is no answer from which the teacher may formulate the next question, the following question may be asked. It may be necessary to draw a square inch on paper to show its size.)

How many square inches, arranged like the squares on a checkerboard, do you think could be drawn on this rectangle? (Pupils will probably suggest to divide the rectangle into square inches and count the number.)

How shall we begin to divide the rectangle into square inches? (This may lead to the suggestion to mark off inches on the four sides and draw the necessary cross lines.)

The class may now count the number of square inches. This, however, does not accomplish the teacher's aim. The pupils must learn a method of finding the number of square inches in the rectangle without counting them. This may be brought out by means of the following diagram.



How many square inches are there in the lower row? (Five.)

How many rows are there? (Three.)

Then how many square inches are there in the three rows, or the whole rectangle? (3×5 sq. in., or 15 sq. in.)

What numbers, then, do we multiply to get the number of square inches in this rectangle?

How shall we find the area of any rectangle? (First find the number of inches (or other unit) in the length and breadth. Then multiply these numbers. The result is the number of square inches in the rectangle, or is the area of the rectangle.)

Notice that two aims are given the class in the above development. The first is to find the number of square inches in a particular rectangle. The second is to find a way of determining the area of any rectangle without dividing it into square units. Before the class is assigned exercises, have the members find the area of another rectangle drawn on the board, without giving them its dimensions.

The area of the above rectangle is not found by multiplying 5 in. by 3 in. It would be equally absurd to say we multiply 5 days by 3 days and get 15 square days. The area of the above rectangle may be expressed in either of the following ways:

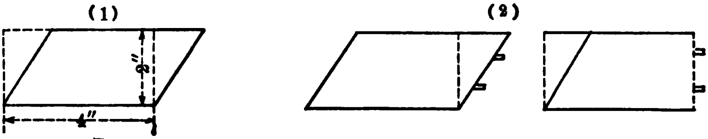
Area of rectangle $= 3 \times 5$ sq. in. $= 15$ sq. in.
or $= 5 \times 3$ sq. in $= 15$ sq. in.
or $= 5 \times 3 \times 1$ sq. in. $= 15$ sq. in.

It is difficult always to use exact expressions. We usually say that the area of a rectangle is found by multiplying the length by the breadth or by finding the product of its two dimensions.

2. *The general parallelogram, or rhomboid.* In order to find a formula for the area of a rhomboid, show first that it may be changed into an equivalent rectangle, as in the diagrams. Diagram (1) on p. 174 shows the use of a drawing or what may be done by paper cutting. The second set of diagrams shows the figures cut out of blocks.

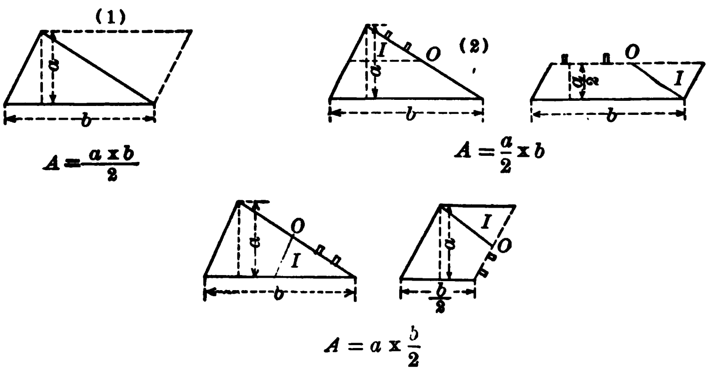
The area of a rhomboid is found to equal the product of one side and the altitude to that side. Have the pupils draw a rhomboid on the board. Measure one side and the altitude to that side and find the area.

Next measure a side adjacent to the first and measure the altitude that corresponds. Again find the area and compare it with the first result.



3. *The triangle.* The area of a triangle is found by comparison with the area of the rhomboid. In the following diagrams one side of the triangle is represented by *b* and the altitude to that side by *a*. The formula may be developed from any one of the three sets of diagrams.

The second and third sets of diagrams show how wooden models may be made and put together. Triangles marked *I* in the first figure of each are turned around the point *O* as a pivot to get the second figure.



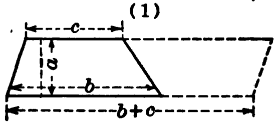
Draw a triangle on the board. Have the pupils find its area by measuring a side and the altitude corresponding

to that side. Do this in three ways by choosing in turn the three sides of the triangle. Compare answers.

4. *The trapezoid.* The area of a trapezoid is found also by comparison with the rhomboid. The second and third sets of diagrams illustrate the use of wooden models.

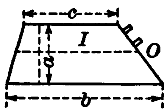
The parts of the trapezoid marked *I* in the second and third diagrams are turned around the point *O* as a pivot to get the second figures.

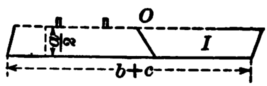
(1)



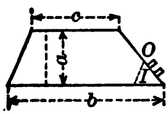
$$A = \frac{a \times (b+c)}{2}$$

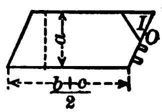
(2)





$$A = \frac{a}{2} \times (b+c)$$

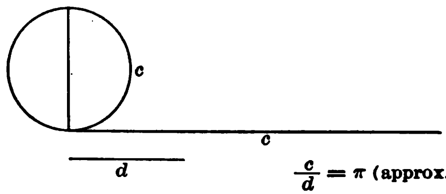




$$A = a \times \frac{b+c}{2}$$

Circumference of a circle. Value of π. In learning how to find a formula for the circumference of a circle, let the class do what the ancients must have done. Measure the circumference of a circle and then its diameter.

Get the ratio between these two lengths. Repeat with larger circles, the larger the circle the more accurate the ratio. Compare results. Since it is difficult to measure the circumference of a circle made with compasses, choose some cylindrical object and measure the circular base. The ratio between the circumference of a circle and its diameter will be found to approximate $3\frac{1}{7}$. A closer approximation is 3.1416. This ratio is commonly designated by the Greek letter π (pi).



$\frac{c}{d} = \pi$ (approx. 3.1416)

We have the formula:

$$\frac{c}{d} = \pi.$$

Hence $c = \pi d$ (in terms of the diameter),

or $c = \pi \times 2r = 2\pi r$ (in terms of the radius).

Pupils should be able to state clearly that in order to find the circumference of a circle without measuring it, we measure the diameter and then multiply the length of the diameter by 3.1416 (or $3\frac{1}{7}$). The same kind of statement should be made in applying all formulas in mensuration.

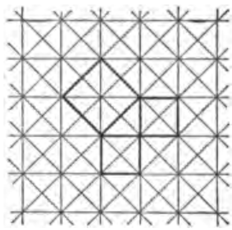
Area of a circle. The area of a circle is usually developed as follows: First draw a number of radii, thereby dividing the circle into sectors. Consider each sector an approximate triangle and find the area of each by multi-

plying the arc by one half the radius. By adding the sectors the formula for the area of a circle becomes $A = c \times \frac{r}{2}$, where c is the circumference and r the radius.

A second formula is obtained by substituting $2\pi r$ for c in the above equation. It is $A = \pi r^2$ (or $\frac{1}{4}\pi d^2$).

It may be necessary in practice to find the area of a circle where it is possible to measure the circumference only. How is the area found?

The Pythagorean theorem. The 47th proposition of Euclid, often called the Pythagorean theorem, is of much practical value. The ancients undoubtedly first discovered its truth in the case of an isosceles right-angled triangle by observing the arrangement of the tiling on their floors. One sees in the accompanying illustration that the square on the hypotenuse of the right-angled triangle is composed of isosceles triangles equal in size and number to those contained in both of the squares constructed on the other two sides of the triangle. The truth of the theorem is also shown in the right-angled triangle whose sides are 3, 4, and 5. (See next page.)

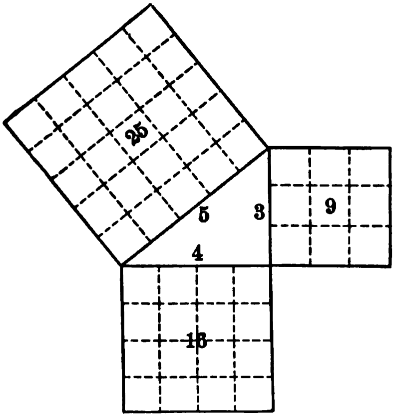


From the figure it is seen that the square on the hypotenuse is equal to the sum of the squares on the other two sides of the right-angled triangle. The numerical relation is:

$$25 = 9 + 16$$

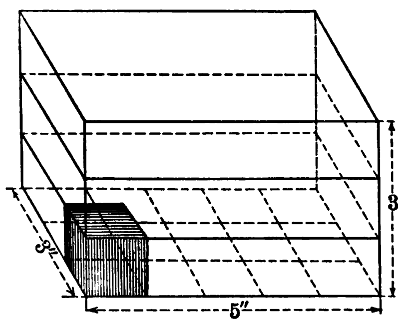
$$\text{or} \quad 5^2 = 3^2 + 4^2$$

The class must learn that the application of this principle consists in finding the length of the hypotenuse when



the lengths of the two sides are known, or in finding the length of one side when the lengths of the hypotenuse and the other side are known, this being done without constructing squares as in the development. In finding these parts in right-angled triangles taken at random the extraction of square roots is required.

Volumes. 1. *The rectangular parallelepiped.* The first solid to be considered in finding volumes is the rectangular parallelepiped. The unit of volume is the cube just as the square is the unit of surface measure. In developing the formula for the volume of the rectangular parallelepiped, use may be made of a model composed of unit cubes or of a drawing like the illustration. Questions like the following may be asked:



How many square inches are there on the base of the parallelepiped? (3 × 5 sq. in., or 15 sq. in.)

Into how many layers has the parallelepiped been divided? (Three.)

How many cubic inches may be placed in the bottom layer? (As many as there are square inches on the base, or 15, for a cube may be placed on each square.)

Then how many cubes are there in the three layers? (3×15 cubes, or 45 cubes, that is, the parallelepiped is composed of 45 cubic inches.)

What numbers have we multiplied together in finding the number of cubic inches in the parallelepiped? (3, 5, and 3.)

The conclusion may now be drawn that in order to find the volume of a rectangular parallelepiped, we multiply the length by the breadth by the height (or thickness). In general terms the formula becomes: $V = a \times b \times c$.

Since $a \times b$ gives the area of the base, we may say that the volume is the base multiplied by the altitude, or height.

2. *Concerning other solids.* If time permits, it is well to develop the formulas for the volumes of the other familiar solids. In any case there should be practice in finding volumes and in finding the superficial areas of the most common solids. Methods of developing the necessary formulas may be found in the higher arithmetics.

Concerning applications. Relate the work in mensuration to land surveying as far as practicable. It is common to find fields in the shape of rhomboids, trapezoids, rectangles, and triangles. The surveying of heights and distances is interesting and profitable.

Textbook problems are not apt to be the most profitable for the class. It is of value for one to be able to solve a problem when the data are supplied, but it is a far better test of one's understanding of a principle to be able to supply the data oneself. Hence be sure that the class knows, for example, what measurements to take in finding the volume of a cylindrical tank. Emphasize this point,

for, if it is neglected, the chief thing to be learned in the applications of mensuration is lost.

Where a course in practical geometry is not given in the grades, its most essential features can be taught in connection with mensuration. It is advisable to go a little out of the way to teach the uses of compasses, scale, triangles, and protractor.

Carpeting, plastering, etc. It is not advisable to spend much time in working problems in carpeting, papering, plastering, and the like. In cities, furniture houses send their men to take the measurements for carpets. Besides there is an increasing use of rugs for floor coverings. Problems in plastering should be worked according to the rules practiced by firms doing that kind of work. Drill is necessary, of course, even in the simplest phases of the work. In carpeting, for example, pupils should understand as a prime essential how to find the number of strips of carpet necessary to cover a floor, but problems concerning matching designs should be omitted. Perhaps the most service that the teacher can render the pupils in the work under discussion is to have them understand when areas are wanted and when volumes. If the pupils make problems from their own measurements they are less likely to figure a room full of plaster than if they use textbook problems.

GRAPHIC REPRESENTATIONS

Value to the citizen. Much use is made of the graph in statistical work, the object being to give the reader ideas of relative value by means of objective illustrations. Any

person who reads the most simple reports of an economic nature must understand graphical representations. We may read the figures giving the strength of the British, German, United States, and Japanese navies, but we are more impressed with the relative strength if the data are put before us diagrammatically.

Representations by lines and other units. A common way of representing statistics of every-day interest is by means of lines so arranged that the eye can easily estimate their relative lengths. Some of the periodicals arouse the interest of the reader by pictorial representations. For instance, cuts of ships of various sizes are drawn to represent the naval strength of various countries. Again, the strength of the standing armies of several countries may be represented by pictures of soldiers drawn to the proper scale for each country.

Representation by a curve. Some statistical data are interpreted in a more suggestive way by means of a curve. A simple illustration, one that may be used in the elementary school, is found in recording temperatures for successive hours of the day. The continuity of the element of time that enters here makes the use of the curve of value.

Value to the pupil. Pupils of the elementary school should be shown graphs like those described above. Good illustrations can be found in a number of journals. Time is well spent in having the class make a few drawings from data which they may be interested in. Such work may well find a place in their study of commercial geography. The computation in drawing magnitudes to the proper scale is a practical study in ratio and proportion.

CHAPTER VII

ALGEBRA AND GEOMETRY IN THE ELEMENTARY SCHOOL

Recent tendencies. The Committee of Fifteen of the National Educational Association recommended, in 1895, that the curriculum of the elementary school be enriched by the introduction of simple algebra and concrete geometry, suggesting that this could be done by eliminating the obsolete and unpractical parts of arithmetic. The effect of the recommendation was generally good. There has been in the last ten or fifteen years a serious effort on the part of teachers and textbook writers to enrich the subject matter of arithmetic. There have been tendencies in certain quarters to go beyond the recommendation of the Committee of Fifteen and to teach algebra somewhat completely in the higher grades, but for the most part it has been used, when taught at all, as a generalized arithmetic. As for geometry, the schools in general have continued only to emphasize its practical features in mensuration. Some schools have made a place for concrete, or observational, geometry, but the movement has not been widespread.

ALGEBRA

Aim and scope. Algebra should be taught in the elementary school for its practical value in the solution of problems and in the application of formulas. The method

of algebra makes clear the reasoning involved in the solution of inverse problems, as has been already shown in discussing the solutions of fraction and percentage problems. The use of the "x" makes the work in proportion simple and comprehensive. In connection with the solution of problems and in using formulas there is need of understanding numerical substitutions and the solution of simple types of equations. There is little practical gain to the pupil of the elementary school in studying algebra beyond these applications. Algebra should not be taught in the grades for the purpose of preparing pupils for the mathematics of the high school. It is different, of course, where the high school algebra takes up the work where it is left off in the last year of the elementary school, as in the case of some city schools. There are a few elementary school algebras that do not make the mistake of trying to cover the whole field of algebra. Such may be profitably used in the grades, but the capable teacher can teach all the algebra necessary in the elementary school by using it to clarify and strengthen the work in arithmetic.

The equation. 1. *First type.* The simplest type of algebraic equation that finds its application in arithmetic is of the form $b+c=a$

or $b-c=a$, where addition or subtraction is the chief operation involved. Using the familiar x in turn for a, b, and c, and numerical values for the remaining letters, this type takes the special forms: $3+2=x$, $x+2=5$, $3+x=5$; $3-2=x$, $x-2=1$, $3-x=1$. Children can readily solve equations like these by inspection as early as

the second grade. A variation of the type arises where a numerical coefficient occurs. Thus we have $3b+c=a$ or $3x+1=7$.

Equations like the last above arise in solving problems of this type:

Ex. A house and lot together cost $5600. If the house cost 3 times as much as the lot, how much did each cost?

Solution:

(*a*) Cost of the house + cost of the lot = $5600

Let x = the cost of the lot

then $3x$ = the cost of the house

Equation (*a*) becomes

(*b*) $3x + x = \$5600$

$4x = \$5600$

$x = \$1400$, the cost of the lot

and $3x = \$4200$, the cost of the house.

2. *Important steps.* The first and most important thing is to get the equation, writing in at first such numerical values as the data of the problem easily afford, as, for example, in equation (*a*) above. Next let some letter such as x represent one of the values to be determined, and from the data of the problem get values in terms of x that will permit equation (*a*) to be written in terms of x and arithmetical numbers, as in equation (*b*) above. Lastly, solve the equation. There should be much practice in writing equations like (*a*) without completing the solution. After the class can determine the preliminary equation in a number of problems, introduce the use of the x and require solutions.

3. *Second type of equation.* Of greater service, perhaps, in arithmetic is the type of equation in which the chief

operation is either multiplication or division. The simplest form of this type is $a \times b = c$

or $$\frac{a}{b} = c.$$

The "x" usually appears as in the following: $3x = 12$, $x \times 4 = 12$, $3 \times 4 = x$; $\frac{x}{5} = 2$, $\frac{10}{5} = x$. The form $\frac{10}{x} = 2$ would hardly occur in arithmetical problems. Children of the third grade can solve equations like these by inspection. A variation of this type arises when other terms are added or subtracted, as in $a \times b + 5 = c$ or $3x + 5 = 7$. We have already discussed the solution of equations of the type $a \times b = c$ and it will not be necessary to repeat it here. (See p. 112 and following, and p. 223.)

Pupils' understanding of principles. While the contention is here made that little algebra is necessary for the elementary school, it is held that the principles involved should be understood. It should be clear that the terms of an equation arising from the solution of a problem must all be alike. In $a + b = c$ the three terms a, b, and c must all be dollars, or cents, or hours, or some other concrete number. The term a cannot be distance and b time and c distance. If this principle is kept in mind, much difficulty will be avoided in forming equations. In $a \times b = c$ there are only two terms, one being $a \times b$ and the others c. If c is dollars, then a and b cannot both be dollars from the nature of the principles in multiplication. If a is dollars, then b in the reasoning becomes an abstract number.

A second principle that is of importance and that must be understood is one which relates to the four fundamental

axioms. In solving equations like those under type 1, use must be made of the addition and subtraction axioms, and under type 2 of the multiplication and division axioms. Pupils need not quote these axioms formally but may say, in solving $3 + x = 7$, "Subtract 3 from both members of the equation"; or, in $3x = 12$, "Divide both members of the equation by 3."

Substituting numerical values. In order to have facility in solving equations and in using formulas, pupils need practice in substituting numerical values in literal equations and expressions. In solving percentage and fraction problems, for instance, the class will need practice in substituting in equations like $r \times B = p$ and then solving in turn for the several letters. If pupils are to be able to make use of formulas, they must have practice in finding the values of expressions like $\frac{a^2 + bc - f}{3ac}$, $\sqrt{ab}$, etc., after substituting numerical values for the letters involved.

Use of formulas. The formulas that occur in the mathematics of the elementary school are simple and are found mostly in mensuration. The equation, $r \times B = p$, used in percentage is essentially a formula. Formulas may also be written for simple and compound interest. Teachers may well give some special attention to the use of formulas. Pupils who later become mechanics and engineers and who often receive no higher education have occasion to use mathematical formulas. Some practice in using formulas should prove of value to the average student. In case some attention is given to the physical sciences in the grades, opportunity is offered for such work. In this

connection we may refer to the frequent inability of pupils who have had a regular course in algebra to solve for unknown quantities where these are not the familiar "x" or "y." Pupils who have "finished" algebra often hesitate when asked to solve for r in $r \times B = p$. They sometimes hesitate in solving for x in $x \times 5 = 13$, where the coefficient of x is not written on the left, or they are unable to solve for d_2 in $p \times d_1 = w \times d_2$. The algebra of the grades can and should give pupils an understanding of such matters.

GEOMETRY

Aim and scope. The algebra of the elementary school is a method; the geometry, chiefly an application. By means of the algebraic method the pupil's understanding of arithmetic is broadened and strengthened. Through his study of geometry he learns of the properties of geometric figures and makes numerical calculations concerning them. He does not take up the subject from the logical point of view until he reaches the secondary school. The practical geometry of the elementary school should be taught experimentally; that is, according to the laboratory mode. A considerable part of the geometry that may be profitably taught in the grades is connected with the subject of mensuration. Pupils may well learn also the properties of the common geometric figures not ordinarily treated under mensuration. In such work opportunities offer for the use of the common instruments of construction and the drawing of figures to scale. There should also be work in simple surveying, especially in relation to finding heights and distances.

Accuracy and neatness in drawings. Accuracy and neatness in construction should be one of the first requirements in any course in geometry, be it in the elementary or the high school. Each drawing should be inspected by the teacher and not accepted until a fair degree of accuracy in measurements has been attained. It is necessary to instruct beginners in some of the most simple matters. Some of these are:

1. Keep one pencil entirely for drawing. It should be of medium hardness and have a long, tapering point.
2. If it is required to connect two points by a straight line, first adjust the ruler to the pencil as it is placed in turn on the two points.
3. Incline the top of the pencil in the direction the line is to be drawn. In other words let the point follow.
4. In drawing a line give the pencil uniform pressure throughout, being careful to begin and end the line abruptly.
5. Where two or more lines meet but do not cross, make a true corner.
6. In drawing a line a specified length, first mark the end points. In locating these points the eye should be directly over the designated mark on the ruler and the pencil held away from the eye at right angles to the ruler and inclined to the paper.

Use of notebook. Drawings and written work should be kept in a special notebook, preferably the loose-leaf folder. All drawings should be on regular drawing filler paper. The page is given a finished appearance by ruling a border line about one half an inch from the edge and from any perforations in the sheet. The first work especially should be most carefully inspected by the teacher. Time will be saved by insisting on neatness and accuracy in the first drawings. Slovenly habits once formed are not easily eradicated.

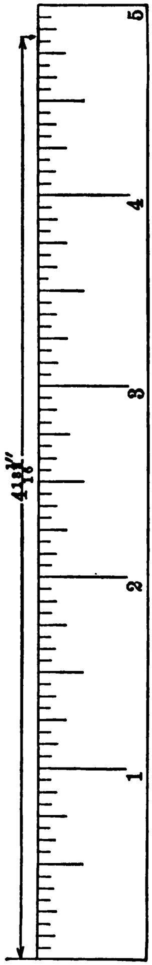
Lettering. Where diagrams are lettered certain standards should prevail. Letter all intersections of lines and isolated points with printed capitals and where single letters are employed to designate lines, use printed small letters. Angles may be represented by numerals, by small printed letters, or by three capitals designating the sides of the angle. All numbers relating to the drawing should be carefully printed. Where measurements are written in the drawing, they should be placed according to the standards of draftsmen. (See drawings on p. 174.) Titles should be printed in capitals. Teachers may, of course, use the style and size of lettering they wish. The following letters and figures are suggested as types:

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz
0123456789

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz
0123456789

Drawing to scale. Almost the first work in a course in practical geometry requires drawing figures to scale. The class should use, if possible, rulers graduated both decimally and to 16ths. The former scale is becoming more common, being used especially in surveying and engineering work. The latter scale is the more generally used and is employed in the manual training work of the school

and in the various branches of the mechanical arts. Work with the decimal scale is of value in connection with the study of decimal fractions.



1. *Use of a ruler reading 16ths.* Let it be required to draw a line to represent a length of 4′ 10″ (4 ft. 10 in.), 1 ft. to be represented by 1 in. and a ruler graduated to 16ths to be used.

4′ is represented by 4″
10″, or $\frac{5}{6}$′, is represented by $\frac{5}{6}$″
Then 4′ 10″ is represented by $4\frac{5}{6}$″.

In order to lay off $4\frac{5}{6}$″ with a ruler reading 16ths, reduce $\frac{5}{6}$ to 16ths. We have $\frac{5}{6} = \frac{13\frac{1}{3}}{16}$. Hence we lay off 4″ and estimate $\frac{1}{3}$ of a 16th beyond thirteen 16ths.

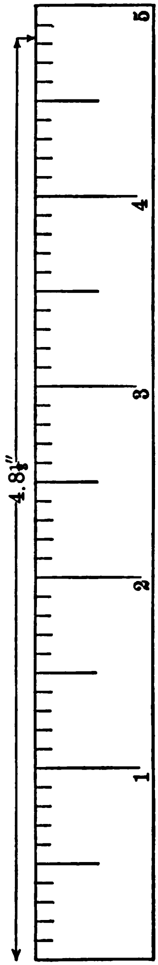
2. *Use of a ruler reading 10ths.* Let it be required to draw the same length of line ($4\frac{5}{6}$″) with a ruler graduated to 10ths. We must change $\frac{5}{6}$ to 10ths. We have $\frac{5}{6} = .8\frac{1}{3}$.

Lay off 4″ and estimate $\frac{1}{3}$ of a 10th beyond eight 10ths. If the ruler is graduated to 20ths, 30ths, etc., reduce the fraction, like $\frac{5}{6}$, to 20ths, 30ths, etc. The numerator gives the number of divisions to be read off.

3. *Another example.* Ex. Make a plan of a rectangular room, 15′ 6″ by 12′ 10″, using a scale of 1″ to represent 3′ (1″ = 3′) and a ruler graduated to 16ths.

COMPUTATION:

Since 3′ is represented by 1″, the number of inches to represent $15\frac{1}{2}$′



equals the number of times $3'$ is contained in $15\frac{1}{2}'$, or $5\frac{1}{6}$. Therefore $15'\,6''$ is represented by $5\frac{1}{6}''$. Similarly, $12'\,10''$ is represented by $4\frac{5}{18}''$. Now change $5\frac{1}{6}''$ and $4\frac{5}{18}''$ to 16ths.

$15'\ 6'' = 15\frac{1}{2}'$.
$12'\ 10'' = 12\frac{5}{6}'$.

We have $5\frac{1}{6}'' = 5\frac{2\frac{2}{3}}{16}$ SIDE WORK: $6\overline{)16}$ $2\frac{2}{3}$

$4\frac{5}{18}'' = 4\frac{4\frac{4}{9}}{16}$ SIDE WORK: $\frac{16 \times 5}{18} = \frac{40}{9} = 4\frac{4}{9}$.

The drawing may now be made with a ruler graduated to 16ths.

Ex. Using a ruler reading 16ths, draw a line to represent a length of $2'\,8''$, the scale being $1'' = 2'$. Draw the same length to the same scale, using a ruler reading 10ths.

Use of models. Models are especially valuable in connection with the mensuration of solids, although they are helpful also in the study of plane figures. (See pp. 174, 175.) Some very good sets are on the market. The pupils may make most of the necessary models themselves. In so doing they get a lasting impression of the geometric figures they are studying and at the same time obtain skill in hand work. Teachers should, however, avoid insisting on undue accuracy in such work at the expense of time which could be better spent in useful applications.

The learning of geometric truths. The learning of geometric truths should be the result of the pupil's self-activity in connection with work of an experimental nature. The method of procedure should provide for drawings or measuring, or both, in order to discover the truths; a clear statement of definitions or principles developed; and the application, either as field work, constructions, or numerical applications. These points are considered in the following development:

Some Constructions and Truths Concerning One Triangle

1. Construct a triangle whose three sides shall each be 3″. (Compasses.)

Definition : A triangle having its three sides equal is called an equilateral triangle.

2. Construct a triangle having one side 1″ and each of the other sides $1\frac{1}{2}$″.

Definitions : (*a*) A triangle having two sides equal is called an isosceles triangle. (*b*) The unequal side in an isosceles triangle is called the base, the angle between the equal sides the vertex angle, and the remaining angles are called the base angles. (*c*) A triangle having no two of its sides equal is called a scalene triangle.

3. Draw an isosceles triangle. Measure the base angles with protractor. What apparent fact do you discover? Is this true for all isosceles triangles? This leads to the statement that the base angles of an isosceles triangle are equal.

4. Draw an equilateral triangle. Measure the three angles. Proceed as in (3).

5. What is the sum of the three angles of the equilateral triangle measured in (4)? Draw an isosceles triangle. Measure the three angles and find their sum. Do the same for a scalene triangle. This leads to the statement that the sum of the three angles of any triangle equals 180°.

6. The vertex angle of an isosceles triangle is 20°. Find each of the base angles. (Do not draw figure.)

7. A base angle of an isosceles triangle is 30°. Find the value of the vertex angle.

Definition : A triangle having one of its angles a right angle is called a right-angled triangle.

8. One of the acute angles of a right-angled triangle is 50°. What is the value of the other acute angle?

9. What is the value of each of the acute angles of an isosceles right-angled triangle? Measure the acute angles of your own wooden triangle.

10. Find the height of a tree or of the school building by using a 45° right-angled triangle.

11. Find the width of a stream by constructing a 45° right-angled triangle.

Field work. 1. *Its nature.* Field work not only gives an interesting application of geometric principles, but may often stimulate pupils to a further understanding of geometry. The simplest form of field work consists in measuring rectangular fields, floor areas, or ground plans of buildings, the object being either to find areas or to get drawing plans. In the surveying of heights and distances greater opportunity is offered for the use of geometric principles, the most important of these dealing with the properties of the right-angled triangle, the isosceles triangle, congruent triangles, and similar triangles. Four things require attention in connection with field work: (1) The taking of observations or measurements, (2) the taking of field notes in a systematic manner, (3) the making of the drawing or plot, and (4) the writing of the accompanying explanation or computation.

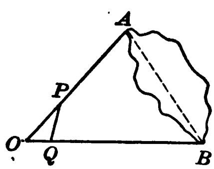
2. *Instruments needed.* For the measurement of rectangular plots, the tape divided decimally or otherwise is alone necessary. In finding by means of congruent triangles distances unmeasurable in the field, the tape (or chain) is the only instrument absolutely necessary. The isosceles right-angled triangle (or any right-angled triangle) and the tape may be employed in finding heights and distances. The transit is, of course, the best instrument for such work, but few schools can afford it, and, besides, there is an educative value in having pupils make instru-

ments for their own use. It is a comparatively simple matter to construct an instrument for measuring angles that will be a fair substitute for a transit. A plane table, which is serviceable for making plots in the field, may be readily constructed. Perhaps the simplest adjunct to the tape is a stick or even a lead pencil. (See pp. 215–217.) A man who wishes to find the height of a tree in order to estimate the amount of lumber in it may do so by lying down at a convenient distance from the foot of the tree, his body in line with the tree, and placing a staff vertically at his feet so that he can see the top of the tree over the upper end of the staff. He then measures the distance from the position of his head on the ground to the staff and to the foot of the tree and finds the length of the staff. From these measurements he finds, by proportion, the height of the tree.

3. *Applications through drawing to scale.* The properties of similar triangles may be profitably studied in connection with drawing triangles to scale and determining the values of unknown lengths and angles.

Ex. Let it be required to find the distance between two points A and B in the field, separated by an impassable barrier.

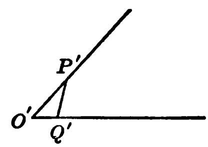
FIELD WORK: Select a point O from which both A and B are accessible. Measure AO and BO. In order to make a drawing that will



give a true picture of the objects in the field, it is necessary either to know the degree value of the angle at O or to know how to construct an angle equal to it. In case no instrument is available for measuring angles in the field, proceed as follows: Locate points P and Q on OA and OB, respectively, and measure OP, OQ, and the tie-line, PQ. Let us suppose, for purposes of illustration, that the following measurements

have been taken: $OA = 120'$; $OB = 144'$; $OP = 12'$; $OQ = 6'$; and $PQ = 9'$. OP, OQ, and PQ are relatively long in the first figure.

Drawing: We wish first to make a triangle to correspond to the triangle OPQ in order to have an angle on paper equal to the angle O in the field. Choose a scale of 1 inch to represent 24 feet ($1'' = 24'$). It follows that OP ($12'$) is represented by $O'P'$ ($\frac{1}{2}''$), OQ ($6'$) by $O'Q'$ ($\frac{1}{4}''$), and PQ ($9'$) by $P'Q'$ ($\frac{3}{8}''$), the triangle $O'P'Q'$ being constructed by means of compasses. Prolong $O'Q'$ so that $O'B'$ equals $6''$ and we have the representation of OB ($144'$). In like manner prolong $O'P'$ so as to make $O'A'$ ($5''$) represent OA ($120'$). Measure $A'B'$, which is found to be $4\frac{9}{16}$ inches. Since 1 inch on paper equals 24 feet in the field, the distance, AB, in the field is readily found.



Note. — The sides of the triangle $O'P'Q'$ in the drawing are drawn to one half the lengths given above.

CHAPTER VIII

THE TEACHER'S PREPARATION OF THE LESSON

Importance of preparation. Every teacher, every day, should plan her lessons systematically, even if the work is mere drill. Where new principles are to be developed, the teacher, especially the beginner, should make a preliminary statement of her plans for the lesson.

Danger of overdeveloping. In presenting new work, inexperienced teachers are apt to overdevelop. Long, tedious presentations of matter that is beyond the appreciation of young minds should be avoided, especially in the lower grades. While it is necessary for the teacher to explain and develop new principles, it is more important to follow this up with sharp, quick drills, for there is no surer way of finding out if principles are understood.

Factors to consider in planning a lesson. The two factors that the teacher should consider in planning any particular lesson are, first, the new principles or phases of subject matter to be introduced, and, second, the special features of the previous work which form a basis for the development of the new. In addition to considering these two aspects of the new lesson, the experienced teacher needs only to provide in advance a list of exercises in which the new principles are applied. The inexperienced teacher needs also to consider in advance two other factors necessary to good teaching, factors which the teacher of experience will

almost intuitively consider in the process of teaching. In the first place it is desirable to give the class the proper incentive to learn the new work, or, as it is commonly called, an aim. In the second place — and this is the supreme test of the teacher's art — the teacher must develop the new principles from those already established by setting up proper lines of association. In other words, she must accomplish the union of the first two factors mentioned above, that of the desired unknown with the known. The five factors to be considered in the lesson are: (1) giving incentives, or aims; (2) preparing for new work through review; (3) presenting new principles; (4) generalizing, or fixing these principles; (5) and applying the principles.

Presence of these factors in different lessons. The five points under consideration may not all be present in any one lesson. The class period often consists of mere mechanical work, such as is common in the earlier grades. It is often the case that new principles which enter into the work may be introduced in such an informal way that the pupils are not conscious of any addition to their knowledge. Nevertheless, the teacher should prepare for this very circumstance and not trust to inspiration alone to guide her in her work. Under many of the topics in arithmetic the teacher will find it possible, and also advisable, to outline her lessons according to the five formal steps in instruction mentioned above. The inexperienced teacher, especially, will find it helpful to organize her work in this way of instruction.

Features of a lesson plan. The following plan, which may be used in presenting new work in any of the school

subjects, lends itself readily to preparatory lessons in arithmetic. The preliminary headings (*a*), (*b*), (*c*), and (*d*) state respectively the general topic under consideration, the special aspect of this topic, the devices and materials to be used in the lesson, and the features of the previous work that bear a close relation to the principles to be established in the work. The title "Method of Procedure" applies to the teaching process in the lesson itself, the development of which recognizes the above-mentioned five steps of instruction.

LESSON PLAN

a. Unit of Instruction: . . .

This may be a topic with or without subtopics. Thus the topic may be Notation and Numeration.

b. Special Phase of the Unit: . . .

This may be a subtopic or a special kind of problem, and constitutes the real title of the lesson. Thus it may be notation and numeration of numbers of two and three figures.

c. Devices and Materials.

A statement should be given of any devices and materials that may be used in the lesson, as bundles of splints and other equipment, to make clear the idea of place value in notation and numeration.

d. Basis for the Lesson.

Give here a summary of the review that bears on the new lesson. Some of this preliminary work may have been done weeks before in relation to other topics and some may have been recently developed as necessary steps

leading to the new lesson. Inexperienced teachers, especially, should thus analyze the new work in relation to the old.

METHOD OF PROCEDURE

(The Day of the Lesson)

1. *Pupils' Aim.*

As the lesson progresses, the class should clearly understand the ends sought. The aims, if there are more than one, may be distributed throughout the lesson, not always formally. Proper incentives given a class at the right time make toward good results.

2. *Preparation.*

Prepare for the new by reviewing the features of the old that are to be used in the development. This has reference to (*d*) above.

3. *Presentation.*

Introduce the new subject matter, relating it to the centers of association set up in (2).

4. *Generalization.*

As a summation of the conclusions reached in (3), deduce the working principle or rule.

5. *Application.*

Test the pupils' understanding of the principles stated in (4) through examples or other forms of applied work.

No rigid adherence to prepared plans. The five formal steps in the above plan cannot be clearly defined in treating some topics. The arrangement given must be con-

sidered as elastic. The pupils' aim is placed first in the above outline, in order to preserve the continuity that exists in the sequence of the next four steps. But in practice it is usually advisable to begin the hour with the review (the *preparation*) and then give the class incentives (*aims*) for learning something new. The statement of the principles introduced in the lesson (*the generalization*) naturally completes the process of developing these principles (the *presentation*). The generalization is logically followed by a testing of the new principles in applied work (the *application*). The application is often interwoven with the preparation and presentation, but it naturally completes the work of the lesson by testing the pupils' understanding of the newly developed principles.

It is not usually possible or advisable to make the phases of the development of the lesson like so many turns in a road. So many unexpected elements enter into the work during the period of instruction that the teacher must necessarily readjust her plans to meet any situation that may arise. She should always abandon any part of a plan when better methods suggest themselves during the progress of a lesson. No two teachers can present a topic in the same way. Neither can one teacher easily duplicate any one of her own lessons as given. Much depends upon the treatment of the preceding material that forms a basis for the new topic. Furthermore, teachers are naturally influenced in their methods of procedure by the texts they are using.

The teacher's aim. The aim of the teacher is usually different from the one given the class, being broader and

sometimes extending through several lessons. For instance, the pupils may be told that they are to learn more about the reading and writing of numbers, so as to understand decimals. The teacher's aim is not only this, but she wants the pupils to understand all about place value. The pupils finally realize this aim in one or more lessons. The teacher's aim is summed up in the generalization, the fourth of the five formal steps in instruction. It is what she wants the pupils to learn.

LESSON I — NOTATION AND NUMERATION

Unit of Instruction: Notation and Numeration.
Special Phase of the Unit: Numbers of one, two, and three digits.

Devices and Materials.

Colored sticks or toothpicks. Rubber bands. A box with three compartments to correspond to the units', tens', and hundreds' orders; numbers to be built on the box lid. In lieu of a box, a diagram may be drawn on a table.

Basis for the Lesson.

a. Ability to read numbers of one, two, and three digits.
b. Ability to write such numbers.
c. Ability to assign number values to groups of objects.

METHOD OF PROCEDURE

(Day of the Lesson)

1. *Pupils' Aim.*

We already know how to write numbers of at least three figures. Let us now make some bundles out of sticks and see what more we may learn about numbers.

2. *Preparation.*

If necessary, review (*a*), (*b*), (*c*) above.

3. *Presentation.*

a. Build numbers on lid of numeration box. Begin with loose sticks in units' box, the other boxes being empty. Have a pupil count out 17 sticks. Place band around 10 of them. Build the number. Build 26, 34, etc., and have them read.

Write these numbers on the board as they are built.
Teach names of units' and tens' orders.

b. Write 29, 37, etc., on the board. Have pupils read and build them.

c. Build, read, and write 124, 236, etc., thereby teaching hundreds' order.

d. Pupils write on the board numbers given by the teacher. Enumerate the orders.

4. *Generalization.*

As a result of the above development the pupils are to learn:

a. The names of the different orders.

b. Any order must contain no number greater than 9.

c. In a number like 326, the 2 represents two bundles or groups, ten units like those of the 6 in each bundle or group.

The 3 represents three bundles of groups, ten units like those of the 2 in each bundle or group. The 3 also represents three hundred of the units like those of the 6.

5. *Application.*

This has been partially given under *Presentation.* Since the pupils can probably already read and write numbers of three digits quite readily, there is no need of continuing this further. Question them on the points emphasized under *Generalization.*

NOTE. — It may be possible to introduce thousands' order in the above lesson. In a later lesson the pupils should learn about the different periods.

LESSON II — SUBTRACTION

Unit of Instruction: Subtraction.

Special Phase of the Unit: Use Addition Method where "carrying" is involved.

Basis for the Lesson.

a. Ability to make change by the making-up method, as commonly used in stores. Very simple examples requiring only mental work.

b. Ability to subtract in an example like $\begin{array}{r} 849 \\ \underline{217} \end{array}$, where the orders in the subtrahend are less than the corresponding orders in the minuend.

c. The habit of subtracting in an example like the above by saying 7 and 2 are 9.

d. Ability to add columns involving "carrying."

METHOD OF PROCEDURE

1. *Pupils' Aim.*

How many can subtract $\begin{array}{r} 2013 \\ \underline{1984} \end{array}$?

We will now learn how to subtract such numbers.

2. *Preparation.*

a. A few written examples based on (*c*) and (*d*) above may be necessary.

b. Quick drill on some of the harder making-up combinations, especially those that will be used before the lesson is over: 9 and what are 17? 8 and what are 13? Etc.

3. *Presentation.*

Let us subtract 93
68.

Is this different from the examples we have been working?

How does it differ?

Does any number added to 8 give 3?

What is the first number larger than 8 that ends with a 3? 8 and what make 13?

Where do we write the 5?

How many to carry? (Why do we say "carry"?)

How many can tell to what figure in the second column we add the 1 carried? (Since the 93 is the sum of the 68 and the desired remainder, we must add the one to the 6.)

We then look at the 6 and think 7.

What do we next say?

What figure do we write to the left of the 5? What is the answer?

Erase the answer and repeat the subtraction.

Next erase the 93 and the line beneath the 68 and add 68 and 25, placing the sum above the 68.

Erase the 25 and repeat the subtraction.

Ex. Subtract 58 from 83.

4. *Generalization.*

a. If, as in the above example, the 8 is greater than the 3, we must think of 13 in the place of the 3. If a 2 were above the 8, we should think of a 12 in its place; etc.

b. Add the 1 "to carry" to the next figure in the subtrahend to the left.

c. After each of these changes are made, subtract as in previous work.

5. *Application.*

a. Write a list of examples on the board and see if the class can follow the directions given in (*a*) and (*b*) above, not necessarily writing down the answers at first.

For instance, write 85
$\underline{27}$. The pupils say: "Think of 15 in the place of 5. Add 1 to 2, thinking 3 in the place of 2."

b. Now give examples for subtraction, requiring statements like the above in each step of the process.

c. Subtract numbers of three or more digits each.

LESSON III — DIVISION OF COMMON FRACTIONS

Unit of Instruction: Division of Common Fractions.
Special Phase of the Unit: The rule for inverting the divisor.

Basis for the Lesson.

a. Multiplication of fractions.

b. Division of fractions with unlike denominators by reducing to common denominators.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

We wish to learn a shorter way of dividing fractions that have unlike denominators.

2. *Preparation.*

Give some examples as in (*a*) and (*b*) above.

Ex. Multiply $\frac{5}{6}$ by $\frac{7}{8}$. Divide $\frac{4}{5}$ by $\frac{2}{3}$.

3. *Presentation.*

a. Divide $\frac{3}{4}$ by $\frac{2}{3}$ just as we have been doing:

$\frac{3}{4} \div \frac{2}{3} = \frac{9}{12} \div \frac{8}{12} = \frac{9}{8} = 1\frac{1}{8}$.

Therefore $\frac{3}{4} \div \frac{2}{3} = 1\frac{1}{8}$.

b. Work this example in multiplication: Multiply $\frac{3}{4}$ by $\frac{3}{2}$. We get $\frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$. Therefore $\frac{3}{4} \times \frac{3}{2} = 1\frac{1}{8}$.

c. Since $\frac{3}{4}$ divided by $\frac{2}{3}$ gives $1\frac{1}{8}$ and since $\frac{3}{4}$ multiplied by $\frac{3}{2}$ gives the same answer, then $\frac{3}{4}$ divided by $\frac{2}{3}$ must equal $\frac{3}{4}$ multiplied by $\frac{3}{2}$, which is $\frac{2}{3}$ inverted.

d. Divide $\frac{4}{5}$ by $\frac{3}{4}$ by reducing to a common denominator. Multiply $\frac{4}{5}$ by $\frac{4}{3}$ and see if the answers agree.

4. *Generalization.*

In order to divide a fraction (or a whole number) by a fraction, multiply the dividend by the divisor inverted.

Thus: $\frac{3}{7} \div \frac{2}{3} = \frac{3}{7} \times \frac{3}{2} = \frac{9}{14}$. We do not need any longer to reduce fractions to common denominators when dividing.

5. *Application.*

a. Work examples of the above type.

b. Work others where either dividend or divisor are mixed numbers.

c. Also choose the dividend a whole number.

LESSON IV — DIVISION OF DECIMALS

Unit of Instruction: Division of Decimals.
Special Phase of the Unit: The divisor a decimal.

Basis for the Lesson.

a. Division of integers, the quotient a decimal, as in dividing 135 by 25.

This necessitates the use of a point at the right of the dividend and the annexing of ciphers.

It necessitates also the proper placing of the first digit in the overhead quotient.

It requires also the placing of the point in the quotient above the point at the right of the dividend.

b. Multiplying a number by 10, 100, etc., moves the point one, two, etc., places to the right.

c. Multiplying both divisor and dividend by the same number does not change the value of the quotient.

d. Multiplying the dividend multiplies the quotient, and multiplying the divisor divides the quotient.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

Ex. If a boy earns $.65 a day, how many days will it take him to earn $7.80? In attempting to divide 7.80 by .65 the class sees the need of learning to divide by a decimal divisor.

2. *Preparation.*

Review as many of the steps in (*a*), (*b*), (*c*) above as may be necessary.

3. *Presentation.*

Ex. Divide 7.80 by .65.

The teacher writes: $.65\overline{)7.80}$

How does this example differ from our previous examples in the division of decimals?

How can we make the divisor a whole number, keeping the same digits? (Multiply by 100.)

How would this affect the value of the quotient?

If we multiply the divisor by 100 so as to get a whole number as a divisor, what else must we do so as not to get too small a quotient? (Multiply the dividend by 100.)

This leads to the statement that we first move the point in the divisor two places to the right so as to rid the divisor of decimals, and then move the point in the dividend the same number of places to the right.

The work stands: $65.\overline{)780.}$

Now place a point above the point in the dividend in the place reserved for the quotient.

The work stands: $65.\overline{)780.}$ with $\dot{}$ above the point: $\begin{array}{r} . \\ 65.\overline{)780.} \end{array}$

Next place the first figure of the quotient, 1, over the proper figure in the dividend, the 8.

The work stands: $\begin{array}{r} 1\;. \\ 65.\overline{)780.} \end{array}$

Write the next figure of the quotient above the 0 after subtracting the 65 from the 78 and continue the division.

A caret may be placed in the new position of the point in both the divisor and dividend. It will be found less

confusing to have the final form as follows, in which the old points are not erased and the new ones are not written in but are imagined to be there.

$$\begin{array}{r} 12. \\ .65\overline{)7.80} \\ \underline{6\,5} \\ 1\,30 \\ \underline{1\,30} \end{array}$$

4. *Generalization.*

We proceed mechanically as follows:

a. Imagine the point in the divisor moved to the right so as to rid it of decimals.

b. Imagine the point in the dividend moved the same number of places to the right.

c. Place the point in the quotient over the new position of the point in the dividend. In short division the quotient and its point are usually written beneath the dividend.

d. Place the first digit of the quotient as in ordinary division.

5. *Application.*

a. Work other examples as above, where the quotient is a whole number.

b. Work examples where the quotient is a decimal.

c. Work examples where ciphers must be annexed to the right of the dividend to accommodate the moving of the point.

LESSON V — PERCENTAGE

Unit of Instruction: Percentage.
Special Phase of the Unit: A first lesson.

Devices and Materials.

Newspaper clippings showing goods advertised at certain per cents off the regular price.

Basis for the Lesson.

a. Aliquot-part relations between decimal and common fractions.

b. Ability to work simple problems in the multiplication of common and of decimal fractions.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

An incentive to learn the meaning of the term "per cent" and to understand its use is created in the *Presentation* below.

2. *Preparation.*

Such parts of (*a*) and (*b*) above as may be necessary.

3. *Presentation.*

A. The class is shown a local newspaper advertisement which may read as follows:

"C. A. Clifford offers holiday goods at $33\frac{1}{3}\%$ off."

Ask what the advertisement means.

This leads to an understanding of the term "per cent" and the use of the symbol %. The terms "per cent" and "hundredths" are interchangeable. We have the following equalities:

$$33\tfrac{1}{3} \text{ hundredths} = 33\tfrac{1}{3} \text{ per cent}$$
$$\text{or } .33\tfrac{1}{3} = 33\tfrac{1}{3}\%$$
$$\text{or } \frac{33\frac{1}{3}}{100} = 33\tfrac{1}{3}\%.$$

Since $.33\frac{1}{3} = \frac{1}{3}$, then $33\frac{1}{3}\% = \frac{1}{3}$.

What, then, does Mr. Clifford mean in his advertisement? *Ans.* That he will sell goods at $\frac{1}{3}$ less than what he usually sells them for.

Suppose Mr. Clifford usually sells a certain class of jewelry for \$36. How much less does he now charge and how much does the purchaser pay?

What would the advertisement mean if it said that goods would be sold at 25% off? 20% off? 30% off?

B. In what other connection have you heard the term "per cent" used? *Ans.* In lending money.

What is meant when a man lends \$200 at 6% interest? *Ans.* That he receives 6% of \$200 as interest.

What decimal does 6% equal? *Ans.* .06

Then what is the interest? *Ans.* .06 of \$200, or \$12.

Much work of this kind cannot be done without a summing up of principles and drill in changing simple per cents to common and to decimal fractions.

4. *Generalization.*

Questions like the following may be asked:

What word may be used instead of per cent?

What two words, then, are interchangeable?

How do we change 40% to a fraction? *Ans.* Say 40 hundredths and write the fraction either in the common or the decimal form.

$$40\% = \tfrac{40}{100} = \tfrac{2}{5}$$
$$\text{or } 40\% = .40.$$

5. *Application.*

a. Give quick mental drills in changing per cents to simple common fractions, as in $12\frac{1}{2}\% = \frac{1}{8}$.

b. Solve problems like the one taken from the above-mentioned "advertisement."

c. Drill on changing simple per cents to decimals, as in 15 % = .15; .07 = 7 %.

d. Solve problems like the one given in simple interest.

NOTE. — The work of the first lesson brings in such simple per cents that it is probably not necessary to give any further direction than that given in the *Generalization*.

LESSON VI — MENSURATION OF THE RECTANGLE

Unit of Instruction: Mensuration.
Special Phase of the Unit: The area of a rectangle.

Devices and Materials.

Drawing of rectangle on the board, divided as described below.

A rectangular board, say 4″ by 9″, marked into square inches. (This may also be used in another lesson in finding the area of a parallelogram.)

Basis for the Lesson.

a. Names of common geometric figures and their chief characteristics, especially points of similarity and dissimilarity between the square and the rectangle.

b. Ability to construct squares and rectangles, using preferably the draftsman's triangles and the scale, or graduated rule.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

How many of you have fathers that own farms? If each father were to sell his farm, how would he calculate

what it is worth to him? This leads to the question of area in general and that of a rectangle in particular.

2. *Preparation.*

Perhaps no special review is necessary, but (*a*) and (*b*) above should be understood.

3. *Presentation.*

Draw on the board a rectangle 8″ by 14″, the longer side, or base, being horizontal.

Mark the sides into divisions of 1″ each.

Draw a horizontal line 1″ above the base so as to form a strip 1″ wide and 14″ long.

Divide this strip into square inches.

How many square inches are there in this strip?

How many strips like this could we draw? Draw them but no more squares.

How many square inches are there in 8 strips? *Ans.* 8 × 14 sq. in. or 112 sq. in.

Into how many square inches, then, may this rectangle be divided?

4. *Generalization.*

Let us erase all the lines drawn in our rectangle. Who can tell, without trying to picture the squares, how we get 112 as the number of square inches that make up the rectangle? (We multiply 14 by 8, or 8 by 14.)

How do we find the number of square inches in a rectangle 16″ by 20″?

Introduce the term "area."

Find the area of a rectangle, the lengths of whose sides are expressed in feet.

The following should be emphasized:

a. The length and breadth must be expressed in like units.

b. The unit of area depends upon the unit of length. (How?)

c. Use of term "dimensions."

d. The area of a rectangle is found by multiplying the number of units of length by the number of units of breadth. Or it is the product of the two dimensions of the rectangle.

e. The term "area" defines the number of square units in the figure.

(Do not say that the area is $5''$ times $4''$, for inches times inches cannot give square inches. The multiplier must always be abstract. Say 5 times 4 equals 20, the number of square inches.)

5. *Application.*

a. Give other examples involving different linear units, requiring only mechanical work. Relate the present work to the use of tables of square measure.

b. Have a rectangle drawn and also its unit of area.

c. Find the area of the schoolroom floor, measuring its dimensions to the nearest foot and half foot.

NOTE. — In the following lessons have plans of irregular figures drawn to scale, choosing figures that can be divided into rectangles. Compute perimeter and area. The ground plan of the school building may give suitable data.

LESSON VII — SURVEYING HEIGHTS

Unit of Instruction: Surveying.
Special Phase of the Unit: Heights.

Devices and Materials.

A measuring tape. A short stick or lead pencil.

Basis for the Lesson.

a. An understanding of the terms: "Vertical line," "perpendicular," "horizontal line," "right-angled triangle," "parallel lines."

b. Ability to make drawings of such figures.

c. Parallel lines are everywhere equidistant.

d. The proportions between corresponding sides in similar triangles. This is not necessary in case the teacher wishes to give these as rules in connection with the lesson, but it is necessary that the pupils be able to solve for x in proportions like $\frac{3}{8} = \frac{6}{x}$.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

How many have heard of lumbermen finding the height of trees when estimating timber? We shall learn to-day how to find certain heights without measuring them.

2. *Preparation.*

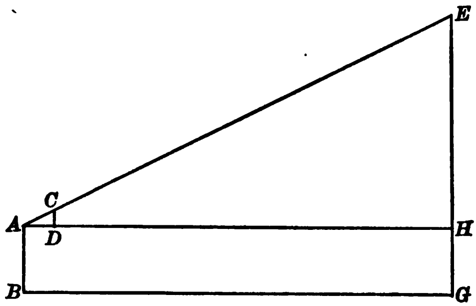
Review the terms given under (*a*) above and such of (*d*) as have been taught.

3. *Presentation.*

We wish to find the height of this room. (Specific aim.)

(As the measurements are taken let a drawing be made on the board, as nearly to scale as possible.)

We shall find the distance from a corner of the ceiling to the corresponding corner of the floor (the line joining the points represented by E and G.)



A pupil stands in the position *AB*, his eye at point *A*, and his arm extended so that when he sights over the top of the stick *CD* held vertically in his hand, he sees the corner of the ceiling (at *E*). At the same time he sights under the lower end of the stick so that the line of sight, *AH*, is horizontal.

The stick *CD* has already been measured, and is 1 foot long. Measure *AB* (4′ 6″), the height of the pupil from his eye to the floor. Measure *AD* (2′), the distance from the eye to the bottom of the stick. Measure *BG* (28′), the distance from the feet of the pupil to the corner of the floor.

Why does *AH* equal *BG*?

Reach the conclusion that since *CD* is ½ of *AD*, *EH* is ½ of *AH*. Therefore $EH = 14'$.

Get the value of *EH* also from the proportion $\frac{AD}{CD} = \frac{AH}{EH}$. Substituting the proper lengths, we get $\frac{2}{1} = \frac{28}{EH}$: Hence $2\ EH = 28$ and $EH = 14$.

How long is EG? $EG = EH + HG = 14' + 4'\ 6'' = 18'\ 6''$.

4. *Generalization.*

a. We measure first the small stick that is held in the hand. Hold the stick in the position described above. Measure the distance from the eye to the bottom of the stick. Measure the distance from the point on the floor where the pupil stands to the point on the floor directly beneath the point whose height is to be found. Measure the height of the pupil from his eye to the floor.

b. See that the pupils can tell how to write the proportion as given under *Presentation*, and that they can finally tell how to get the required height.

5. *Application.*

a. Substitute values in another drawing on the board to test (*b*) above.

b. Find another height to test (*a*) and (*b*) in *Generalization* either in this lesson or the next.

c. Have the class draw to scale the figure represented under *Presentation* for home work.

NOTE. — CD and AD in the above figure may be in inches and BG (or AH) in feet.

It is not necessary that the bottom of the stick or pencil be held on a level with the eye, but this avoids a little complexity.

The use of the 45° right triangle provides an easy method of finding heights.

LESSON VIII — KEEPING ACCOUNTS

Unit of Instruction: Keeping Accounts.
Special Phase of the Unit: First lesson in keeping a personal cash account.

Materials:

A small cash book, about $3\frac{1}{2}''$ by $6''$.

A ruler. Black and red ink. Pens for each kind of ink.

Basis for the Lesson.

a. Accuracy in column addition.

b. Ability to use the addition method of subtraction in writing down a missing addend in a column of figures whose sum is set down. (This ability may not be required, although the addition method of subtraction makes the work relatively simple.)

c. Multiplication of decimals as related to United States money.

d. Possibly multiplication of easy common fractions.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

How many of you know how much money you spend each week? We shall learn to-day how to keep account of our expenditures.

2. *Preparation.*

On the day previous the pupils are asked to bring in a short list of amounts of money received by them on specified dates, within the last week or month. Fictitious transactions may be used to supplement the list. They should also bring in a list of amounts (real and fictitious) paid out by them on specified dates within the same period.

3. *Presentation.*

Let the children contribute as much as possible to the development of the lesson.

a. What title is to be written in the front of the book? (——'s personal cash account.)

b. On which page of the open book do we write the list of expenditures? (The credit, the right-hand, side.)

Why do we credit cash that is paid out? *Ans.* Because the person paying the money must give himself credit for doing this, in his own cash account.

c. Why do we debit cash that is received? *Ans.* Because the person receiving the money is indebted to that extent to some one for rendering him that service.

d. Show where to write the year, the day of the month, the terms Dr. and Cr., and the items brought to class as assigned the previous day.

e. If double columns are used, write original entries in the first of the double columns on each page. Write sub-totals, balances brought down, amounts carried forward, and amounts brought forward in the right-hand columns.

f. Plan to have balance made near the middle of the first page. Balance the account by the method suggested in (*b*) of *Basis for the Lesson*, p. 218.

g. Write other entries and write at the bottom of each page the amounts to be carried forward.

h. At the top of the second open page write the amounts brought forward. Enter other items.

i. Rule all lines in red ink. Write balance in red ink, but not the balance brought down on the Dr. side.

4. *Generalization.*

This is largely provided for in the above. But emphasize especially:

a. The significance of debiting and crediting entries.

b. The object in balancing an account. The significance to be attached to the amount written on the Cr. side that balances the account.

c. The significance of "bringing down" the balance.

5. *Application.*

Continue the work already begun.

Note. — The double-column page is not necessary, but its use is advisable after the single column has been employed. The meaning of debiting and crediting may be deferred to a later lesson.

In a later lesson teach the writing of receipts and receipted bills in relation to some of the entries.

Pupils may do a buying and selling business among themselves, using imitation money and small printed cards to represent various kinds of merchandise. Such money and cards may be purchased from publishing houses.

If any business is done on credit, open accounts with the proper persons, thus introducing the use of a ledger.

LESSON IX — BONDS

Unit of Instruction: Lending and Borrowing Money.
Special Phase of the Unit: Bonds.

Materials and Devices.

School, municipal, corporation, or United States bonds, if available. Otherwise a blank form or copy of a bond.

Basis for the Lesson.

Simple interest. Promissory notes. Ordinary applications in business arithmetic.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

The teacher naturally tells the class that they are to learn something about bonds. A definite aim is developed in the *Presentation.*

2. *Preparation.*

A preliminary review is perhaps unnecessary.

3. *Presentation.*

The city of Chico, California, decided, in 1902, to install a sewer system to cost $45,000. No funds were available. In what two ways could the money have been raised? *Ans.* By special tax and by borrowing.

It was decided to borrow the money. When an individual borrows money, he gives the lender a promissory note. Cities that borrow money give bonds instead of notes. The teacher should here bring out the points of similarity and dissimilarity between notes and bonds.

After the borrower determines the rate of interest he is willing to pay and at what intervals he wishes to pay back portions of the sum borrowed, he usually advertises the bonds "for sale." The highest bidder is the person who gives the borrower the greatest "bonus" for the privilege of lending him the money at a satisfactory rate of interest. This bonus, or gift, to the borrower is called the premium. The borrower issues as many bonds as there are times that he pays back portions of the principal. A bond is said to be redeemed when it is returned to the borrower on his paying the principal named in the bond together with the interest due.

4. *Generalization.*

a. The pupils learn that the bond has the function of a promissory note.

b. They learn how cities (and other corporations) borrow money and how the money is paid back.

c. They learn why the lender sometimes pays premiums on bonds.

5. *Application.*

Ex. The City of Chico, California, on Jan. 1, 1902, issued bonds for $45,000 to provide funds for installing a sewer system. The bonds were eighty in number and of equal face value. Two were to be redeemed on Jan. 1 of each year, beginning with Jan. 1, 1903. Interest was to be at the rate of 5% per annum, payable semi-annually on Jan. 1 and July 1 of each year. The bonds were sold at a premium of $1100. Find the total amount that will have been paid by the city when all the bonds have been redeemed, and find the net cost to the city.

LESSON X — ALGEBRAIC FORMULAS

Unit of Instruction: Algebraic Formulas.
Special Phase of the Unit: Solution for the unknown in equations of the type, $rB = p$.

Basis for the Lesson.

a. Use of the algebraic method in equations of the type, $2x + 3 = 9$, although this is not absolutely necessary.

b. Ability to solve for the unknown in equations like $5c = 10$ is necessary.

METHOD OF PROCEDURE

1. *Pupils' Aim.*

The class has probably experienced difficulties in solving inverse problems in fractions and in percentage. Such problems will now be readily understood by the algebraic method.

2. *Preparation.*

Solve: (*a*) $5c = 10$; (*b*) $8 = 4d$; (*c*) $\frac{2}{3}x = 12$.

In examples like these, pupils have already learned to divide both sides of the equation by the coefficient of the unknown quantity.

For example, in solving $\frac{2}{3}x = 12$, we have $x = 12 \div \frac{2}{3} = 18$.

3. *Presentation.*

Consider the equation $r \times B = p$. The sign $\times$ is usually omitted. Thus we have $rB = p$, in which either r, B, or p may be taken for the unknown quantity.

(a_1) Let $r = 5$ and $p = 10$. Solve for B.

SOLUTION:
$$\begin{aligned} rB &= p \\ 5B &= 10 \\ B &= 2. \end{aligned}$$

(a_2) Let $r = \frac{2}{3}$ and $p = 12$. Solve for B.

SOLUTION:
$$\begin{aligned} rB &= p \\ \tfrac{2}{3}B &= 12 \\ B &= 12 \times \tfrac{3}{2} = 18. \end{aligned}$$

(b_1) Let $B = 8$ and $r = 5$. Solve for p.

SOLUTION:
$$\begin{aligned} rB &= p \\ 5 \times 8 &= p. \\ p &= 40. \end{aligned}$$

(b_2) Let $B = 12$, and $r = .33\frac{1}{3}$. Solve for p.

SOLUTION:
$$\begin{aligned} rB &= p \\ \tfrac{1}{3} \times 12 &= p \\ p &= 4. \end{aligned}$$

(b_3) Let $B = 16$ and $r = 12\frac{1}{2}\%$. Solve for p.

SOLUTION: $$rB = p$$
$$\tfrac{1}{8} \times 16 = p$$
$$p = 2.$$

(b_4) Let $B = 8$ and $r = 21\%$. Solve for p.

SOLUTION: $$rB = p$$
$$.21 \times 8 = p$$
$$p = 1.68.$$

(c_1) Let $B = 18$ and $p = 3$. Solve for r.

SOLUTION: $$rB = p$$
$$r \times 18 = 3$$
$$r = \tfrac{3}{18} = \tfrac{1}{6}.$$

(c_2) Let $B = 24$ and $p = 9$. Solve for r in terms of per cent.

SOLUTION: $$rB = p$$
$$r \times 24 = 9.$$
$$r = \tfrac{9}{24} = \tfrac{3}{8} = 37\tfrac{1}{2}\%.$$

4. *Generalization.*

The principles were emphasized in the presentation and may not need further emphasis here.

5. *Application.*

Continue mechanical drills like those in the presentation. The class should preferably be at the board, the formulas being always written down. As the teacher gives the values to be substituted, the pupils write them in. The drill at first may be on mere substitutions without solutions.

As a further application the teacher may give simple fraction and percentage problems, the class promptly substituting the proper values in the formula.

Ex. A man sold a lot at a gain of $16\frac{2}{3}\%$ on the cost. If the gain was $200, what was the cost?

NOTE. — The class may also solve in turn for B and r in $rB = p$ and then substitute the numerical values given in (a) and (c) above.

CHAPTER IX

IN RELATION TO SCHOOLROOM PRACTICE

MODES OF INSTRUCTION

WE shall follow the practice of some of our recent books in using the term "mode" with respect to the manner of conducting class work and the term "method" with respect to the treatment of subject matter.

Statement of modes. The modes of instruction in the elementary school may be classified into the recitation mode, the developmental mode, the heuristic mode, the class mode, the individual mode, and the laboratory mode. There is considerable overlapping in this classification. It will suffice here to describe briefly the essential characteristics of each mode.

Recitation mode. The recitation mode requires that the pupil tell what he knows of previously assigned work. The element of the quiz or examination enters here. This mode is illustrated in arithmetic where the class is required to explain at the board or from papers problems previously given for home work. The recitation mode is typically American. In the schools of Germany, France, and other European countries, where the assignment of home work is minimized, this plan of instruction is little used.

Developmental mode. In partial opposition to the recitation mode is the developmental mode, in which the

teacher brings out by proper questioning the new phases of subject matter. Pupils complete, outside of class, the work begun in class. The developmental mode has a marked inductive character, whereby general principles are established by the proper relating of known particulars. This mode is more common in Europe than in this country.

Heuristic mode. Where the developmental mode is used to the extent that the pupils work out new principles for themselves, the mode of instruction is called heuristic. The heuristic mode is real or false according as the teacher causes the pupils to develop the principles or merely makes them believe they do.

Class mode. In using the class mode the teacher seeks to have all the class participate in some particular phase of the discussion or work. Each one is given an opportunity to express an opinion or offer a suggestion. All pupils are expected to cover the same amount of subject matter at the same rate.

Individual mode. The opposite of the class mode of instruction is the individual mode. The individual pupil is here the center of the teacher's interest. Instead of trying to develop the powers of all the class alike, the teacher recognizes the various capacities of the pupils and arranges their work accordingly, both with respect to difficulty and amount.

Laboratory mode. The laboratory mode may be defined as the union of the developmental and individual modes in connection with some sort of laboratory equipment. The correlation of mathematics with the physical sciences offers an opportunity for the use of this mode. This correlation

may be made even in the elementary school. The teaching of the metric system in connection with the work of the physical laboratory illustrates an instance. The laboratory mode may be used in teaching decimals, where the teacher gives the work an experimental character by the use of decimal measures; for example, the metric rule, the decimal tape, and coins in the United States money. The laboratory mode is also used in the business arithmetic of the upper grades, where the pupils create "business" and make the necessary transactions.

Choice of modes. The lines of distinction between the above modes of instruction are more marked in the teaching of the secondary than in the elementary school subjects. While the teacher of arithmetic will not confine herself to any one mode, but will use whatever seems best in any particular lesson, yet she should not fail to have system in the routine work from day to day.

The work of the early years is adapted to the use of the developmental mode. Even the heuristic mode has a place, where "discoveries" are made in connection with measuring and other forms of sense-training work. The teacher must remember in using the developmental mode in the primary grades that there is danger of overdeveloping principles, for young children are not concerned very much with the "why" in arithmetic. There is need in much of the work of employing what may be called a didactic mode, in the use of which the pupil is told what to do and the teacher sees that he does it. The developmental mode has a place in the higher grades, where it is of value in establishing new working principles.

Mode influenced by method. The mode of conducting the work of the class hour is largely dependent upon the method chosen in the treatment of subject matter. Corresponding to the inductive method in the treatment of subject matter is the inductive mode, or, as it was called in a previous connection, the developmental mode, in which the teacher causes the movement of the lesson to follow the spirit of the inductive method of reasoning. The class and recitation modes of instruction have little of the inductive element in them, but much of the deductive method. The heuristic and laboratory modes, by virtue of their relation to the developmental mode, depend upon an inductive treatment of subject matter.

DETAILS OF ROUTINE WORK

There is need of system in every phase of the teacher's work. It is perhaps in no way more essential than in connection with the details of the routine of class work.

Mental and written arithmetic. There should be provision for special work in mental arithmetic. As a rule, the first five or ten minutes of the period should be given to this work. It serves both as a review and as a means of getting the class mentally awake. Care should be taken to have the pupils use mental arithmetic in connection with written arithmetic as a means of shortening the work. This matter should be watched especially by teachers in ungraded schools where it may be necessary to drill in mental arithmetic whenever the program permits.

Board and seat work. There should be much opportu-

nity for board work. The brighter pupils like it and the weaker pupils need it. Where the class or group mode of instruction is used, an opportunity is offered for systematic drill work in connection with board work. Errors are easily discovered and correct forms are put before the eyes of the pupils. Seat work gives opportunity for the pupils to work without the aid of the teacher. The teacher should keep a record of all such independent work done at the seat. In case the class is large it will be advisable to divide the class into two sections, one for board work and the other for seat work, the two alternating.

Home and school work. In giving home work care should be taken not to assign more than the class can do unaided and well. When too much is given them, the pupils usually get aid from others whose methods are often at variance with the guiding principles laid down by the teacher. An excessive amount of home work also leads the pupils to slovenly habits in the matters of form, which in turn lead to habits of inaccurate thinking. The teacher should not assign any more home work than she can carefully inspect. Practically no home work should be assigned in the earlier grades.

Home work should supplement school work and should serve as a test for determining whether the principles already developed are understood. It should be especially valuable for drill in applications. School work should provide for rectifying any errors found in home work and for a further development of subject matter.

Work with and without explanation. Provision should be made for the explanation of work, both orally and in

written form, and for mere mechanical work in which the aim is to get the answer by the shortest method possible. The latter sort of work should predominate in the lower grades and should receive equal emphasis with the former in the upper grades. Problems and mechanical processes should be explained in order to fix principles. When the principles are once established, explanations are no longer necessary. A further discussion of the explanations of work will be found on pp. 123–127.

Form of written work. Pupils should receive definite instructions regarding the form of written work. Teachers are in common agreement with respect to the form of the written work involved in all of the fundamental operations in arithmetic. In relation to the written forms in the solutions of problems there are variations in matters of detail. These details should be worked out in any school or system of schools so that the pupils will not find any inconsistencies in these matters as they pass from grade to grade. It may not be out of place to emphasize here the need of unity in all matters of detail as they relate to the formal work of the pupil in his progress through the grades.

Some of the formal things to consider in written work are:

The placing of the pupil's name.

The numbering and arranging of the examples on the page.

The tabulating of concrete numbers in the solution of problems.

The placing and labeling of the answer.

The arrangement of the main steps in the solution and the placing of any side work. All work that is not mental should be on the paper handed in. The practice of doing side work on scratch paper tends toward slovenly habits in matters of form and hence toward inaccurate thinking.

Care in the alignment of rows and columns and in the formation of figures.

Marking papers. The best systems of marking papers are usually the simplest. Many teachers mark by per cent, but this plan has several disadvantages. It is hard to grade within one per cent of the correct grading, and pupils are not usually satisfied with such close estimating. The plan of marking "excellent," "good," "fair," and "poor" seems to give the most satisfactory results. These terms have a meaning both to pupils and teacher, and there are no hairsplitting differences to consider. It is true, however, that the teacher may find it convenient to keep her own records in per cents on account of clerical advantages. In primary work, it is common to write the number of examples correct under the pupil's name or the number worked and the number incorrect.

1. *Answers right or wrong.* Examples of a mechanical nature should be marked wrong if there is any error in computation. The same standard should obtain in problem work after new principles are once established. In adding columns of dollars and cents, a mistake of one unit in the cents' column makes little difference in the answer, but an error of one unit in the thousands' column would be

very serious. Obviously the example should be marked wrong in either case.

2. *Indicating errors.* Whenever possible, the teacher should indicate errors in the work and let the pupils find out the special difficulties and make the necessary corrections. When pupils are permitted to pass over errors without correcting them, one of the main purposes of the daily written work or test is lost.

SUGGESTIONS REGARDING CERTAIN FORMS OF EXPRESSION

The use of incorrect or inaccurate expressions on the part of the teacher often causes misconceptions in the mind of the pupil. Most of the suggestions that follow relate to common errors of expression.

1. Say "naught" for the symbol 0, not "aught." Aught means anything. Naught means no thing, or nothing.

2. Say 2 and 4 are 6, not 2 and 4 is 6. We consider the subject as plural and hence it requires the plural form of the verb.

3. In the form $2+4=6$, say 2 plus 4 equals 6. The singular form of the verb is here used because we consider the $2+4$ as an expression, such as is used in algebra.

4. In the early work the pupil may write 3 cents or 3 ¢, but he should soon learn to use the standard form $.03.

5. In repeating the multiplication tables, it is better to say

one 3 is 3		once 3 is 3
two 3's are 6	although	two times 3 is 6
three 3's are 9		three times 3 is 9

is commonly used. The first form relates closely to serial counting, upon which the learning of the tables is usually based.

6. Either of the written forms

$$\begin{array}{lcl} 2 \times 3 & \text{or} & 3 \times 2 \\ 3 \times 3 & & 3 \times 3 \\ 4 \times 3 & & 3 \times 4 \end{array}$$

may be used, but the first is to be preferred for the reason expressed in (5).

7. Either of the forms 2 × 6¢ or 6¢ × 2 may be used. The first is to be preferred in case the suggestions in (5) and (6) are adopted. The form 2 × 6¢ is read "2 times 6¢" and the form 6¢ × 2, "6¢ multiplied by 2."

8. In division, do not say "goes into," but "is contained in."

9. In referring to the fraction $\frac{2}{3}$, say two thirds or 2 divided by 3, not 2 over 3. It is well to use contracted forms, but forms that are not contracted and do not convey any meaning should be avoided, at least in the elementary school.

10. In reading whole numbers do not say "and." In reading mixed numbers use "and" only to connect the whole number and the fraction. Thus we read 326, three hundred twenty six; $9\frac{4}{5}$, nine and four fifths; and 326.456, three hundred twenty-six and four hundred fifty-six thousandths. Business men use the "and" unrestrictedly, but pupils can write numbers with less confusion if the teacher follows the above suggestion. In writing decimals, for example, it helps to know that "and" is read only at the decimal point.

11. Avoid errors like $\frac{2}{3}$ = \$10 and 8% = 40, where the meanings are that $\frac{2}{3}$ of a certain amount of money equals \$10 and that 8% of some number equals 40. Such errors lead to inaccurate thinking in problem work.

12. Do not write $\frac{1}{6}$ of \$12 = \$2 + \$9 = \$11. Errors like this and the one just pointed out in (11) arise out of a disregard of the meaning of equality. Pupils should be encouraged to see that the total of all terms on the left of the sign equals the total on the right. The error probably arises from carelessness in oral explanations.

13. We annex, not add, zeros to the right of a number.

14. Do not say 3 times greater for 3 times as great, 3 ft. square for 3 sq. ft., nor two last for last two; and *vice versa*. Illustrate the correct usage of each of these six phrases.

CHAPTER X

SUPERVISED TEACHING OF ARITHMETIC IN TEACHERS' TRAINING SCHOOLS

Author's aim. It has been the aim in the preparation of this chapter to give practical help to the student teacher in normal training schools or any schools where students teach arithmetic under supervision. The teacher without experience and without the advantages of professional training may, it is hoped, get some helpful suggestions. It has been thought wise to present this material in the form used by the author in conference work with his own student teachers. This will account for some of the explicit directions given.

Specific directions necessary. Inexperienced teachers need specific instruction and supervision in the details of their work just as their pupils in turn will need careful directions in the most commonplace matters. It does not always follow that a student teacher will properly handle without aid a special method recommended for her use. Teachers, like other persons, are apt readily to go back to earlier habits. It is very easy to teach something the way it was taught us in our earlier years. Again, directions concerning routine matters of class work and management cannot be too carefully considered. If possible, these directions should be in written form in order

that they may be readily accessible to student teachers. While the material given below has been organized with special reference to the teaching of arithmetic, matters of general teaching value are necessarily included. The material offered represents the most essential things that the author has tried to impress upon student teachers in arithmetic under his charge.

Perhaps a brief statement of the relation between teaching practice and normal instruction in the author's own school may help to a better understanding of that which follows. All normal students are required early in their two years' professional course to take a preliminary five weeks' course in the mechanical processes of arithmetic and in simple problems. Students found to be deficient are remanded to an elementary course in arithmetic. Before teaching in the training school, all students take a ten-weeks' course in arithmetic and methods, and, while teaching, for ten weeks, they take a parallel course in arithmetic and methods, the class hour being used also for conferences on problems arising from teaching arithmetic in the training school. Conferences are held also with individual student teachers as occasion demands.

OBSERVATION OF CLASSES

Value of observation. It is possible that some good may be gained from the most perfunctory observation, but if the time thus spent is valuable the student should not attempt to visit a class for purposes of observation without having a conception of definite things to be ob-

served. A report on such haphazard observations is usually barren and worthless. If, however, the student teacher is advised of the things that should be reported upon after the observation, much real help may be derived from it.

General Instructions regarding Observation

1. Be in the classroom before the recitation begins. Avoid being conspicuous.

2. Do not take notes during observation. You may confuse the teacher.

3. As soon as possible on leaving the class write down your impressions. Later, classify them according to the list that follows, using the same numbers for your answers, but not copying the questions in your report. Avoid a mere "yes" or "no" in your answers. Let your report show that you have been watchful and that you understand the import of the questions.

4. Observe for one week the class (identical pupils) you are to teach. For instance, if you are to teach the new Low Fourth, observe the finishing High Third. Make it a special point to learn the names of your prospective pupils.

5. Report to your supervisor on the day of your first observation, so that you may be the better prepared for your second observation. If the first report is satisfactory, the others may be made supplementary to the first and may be given verbally.

6. Observe one or more classes other than the one you are to teach, preferably the grade above and the grade below.

7. As further preparation for your work, read the course of study in arithmetic, especially the work of your own grade.

8. All teachers are expected to make flash cards and number charts for present and future use. Those who need these for present use should have them made before taking charge of their classes. Teachers of the first two grades will need flash cards for teaching pupils to recognize the individual number symbols and charts for teaching them to recognize the place of numbers in the number series, 1–50 and 1–1000. First-grade teachers should also have cards with number groups (spots) on them. Teachers of the third grade will need flash cards for teach-

ing the addition and subtraction combinations. Teachers of the Fourth Grade will need flash cards for teaching the multiplication and division tables. Third-grade teachers should also have the number charts just mentioned. Large rectangular charts on which are written single columns will assist in the teaching of column addition by saving time used in writing the work on the board. Samples of cards and charts together with directions for making them are provided for the student teachers.

Specific Instructions regarding Observation

1. Was the room well ventilated, lighted, and heated? Were the desks, blackboards, and floor in good order?

2. What system had the teacher when sending groups of pupils to the board?

3. Did they arise quickly and stand erect and away from the desk when reciting?

4. Had the teacher materials for use in readiness, including any work on the board?

5. What was the plan of collecting and distributing papers and materials?

6. How were pupils at the board kept from copying?

7. Was the work varied to prevent pupils from getting tired or losing interest?

8. Had the teacher planned her lesson with sufficient care?

9. Were there evidences of unfamiliarity with subject matter on the part of the teacher?

10. Was the teacher especially alive and resourceful?

11. Was the class interested and full of the work spirit? Why?

12. Was the lesson developmental, review, or drill?

13. What proportion of time was given to oral and written work? What proportion, to board and seat work?

14. What was the nature of the work of the first few minutes?

15. What made the teacher successful (or not) in handling objects and materials in instruction?

16. Did the teacher get the entire attention of the class at the beginning of the period or whenever she desired it? How?

17. What specific acts of the teacher made toward good discipline? What was the cause of any poor discipline?

18. Was there effective drill? What was its nature?

19. How was the backward pupil reached?

20. How was the teacher's questioning made effective? Was there any system in naming the pupil after the question was asked?

21. Were all pupils reached and kept busy?

22. Did the pupils use good or bad oral and written form (both in language and in mechanical work)? Were there evidences of good instruction in this matter on the part of the teacher?

23. Did the teacher consider the point of view or comprehension of the pupils or were their interest and capacity unduly taxed?

24. Did you note any time-saving devices?

25. Other comments. (Read "Matters Concerning the Work" in the instructions to teachers below.)

ON TAKING CHARGE OF CLASSES

1. Be natural, yet dignified. Do not try to look like a "school-teacher." Pupils may form a correct estimate of your personality in the first five minutes. If you have a positive feeling that you have the situation in hand, the matter of good discipline is probably solved. Have confidence in your own ability because you know vastly more than do the children whom you are to teach.

2. Introduce yourself to your pupils.

3. Attend to matters of light, heat, and ventilation. Use daily care in these matters.

4. Learn the names of your pupils.

5. Assign the text and materials to be used bv the individual pupils.

6. See that the room is provided with necessary supplies, such as crayon, erasers, and pointer.

7. Discuss the previous work with the class.

MATTERS CONCERNING THE WORK

1. When the instruction requires special materials like blocks, liquid measures, rulers, etc., have them ready before the class hour. Materials and supplementary texts are in your supervisor's office.

2. Save time by having necessary work written on the board.

3. In the first two or three grades, have the board spaced for the children with vertical lines and names above. Have a short piece of crayon for each pupil and an eraser for at least each alternate pupil. In

the other grades let pupils prepare their own spacing, writing their names above.

4. Have an orderly method for groups of children to pass to and from the board. Use the commands of face, arise, pass. Do not have pupils arise in rapid drill work.

5. Use special care in matters of attention and in the execution of commands with the younger children. Pencils and crayons are not to be used until the teacher so directs. Devices like placing hands in the lap will for a time help keep children attentive.

6. All boards and the floor beneath should be left clean.

7. Insist on neatness in all work. Also require good use of English. Have examples well spaced and figures well made.

8. Have top and side margins in written work. Assign places for the pupils' names and the day or date. Instruct how to fold if papers are to be folded.

9. Save time in collecting and distributing papers and materials by having them passed up the line or having special pupils assist.

10. Prevent copying in board work by giving pupils prepared slips with one or more examples on each. Interchange slips. In any case do not assign the same example to adjacent pupils.

11. Return papers to pupils for correction.

12. Give weak pupils individual help, outside of class, if necessary.

13. Provide enough work. Do not allow pupils to finish before the end of the period. Do not expect the weaker pupils to do as much as the stronger ones.

14. Development work with objects or in new subject matter in general should be done with sufficient quickness to prevent dawdling. A teacher who gets pupils readily to apply a principle usually has spent much time in her own preparation.

15. Do not give pupils help when they are supposed to work independently.

16. Do not keep the younger children too long at one thing. They tire easily.

17. Make short assignments of work in connection with new topics.

18. Do not underestimate the value of reviewing the most elementary principles in the more advanced work.

19. Do not forget the importance of much oral work. In the intermediate and upper grades begin the recitation period with quick mental drills. The work of the first two years is practically all oral.

20. Do not speak disparagingly of any of your pupils or of your class. If pupils do not do well, it is probably your fault.

21. Let pupils know that you appreciate their efforts.

22. For your own improvement, study carefully the course of study in arithmetic, re-reading especially the introduction, the work of your own grade, and the work of the grades immediately preceding and following your grade. Remember that the best teacher in arithmetic is the one who has natural teaching ability, tact, and sympathy for children combined with a thorough knowledge of the subject matter of arithmetic. A teacher with native mathematical powers but who has little regard for the art of teaching is a poorer teacher than the one who has only fair mathematical attainments but an abundance of enthusiasm and love for her work.

REPORTS AND OTHER MATTERS

Toward the last of each week, student teachers should confer with their supervisor regarding the approximate work to be covered the following week. An outline like the following is to be filled out and filed by 8.30 each Monday morning. Notice that the upper half of the form shows the work covered during the week preceding and the lower half that planned for the following week. The chief purpose of these reports is to furnish ready information to supervisors and to substitute or new teachers. Where a class is divided into groups, report on each group. Mimeograph copies of this outline may be obtained from the supervisor.

WEEKLY OUTLINE FOR ARITHMETIC TEACHERS

Grade -------------- Teacher --------------

Work Accomplished for Week Ending -------------------

(Give sufficient detail so that a new teacher could easily take up the work. Refer to pages of the text and supplementary books.)

Review given in the formal processes -------------------

Review in concrete problems. Language forms and explanations required ---------------------------------------

Special phase of the unit of instruction in new work given ---

Devices, special drills, or helpful objective work used --

In what lines has there been noticeable improvement? Additional remarks --

What lines need strengthening? ---------------------------

Work to be Accomplished in Week Beginning -------------

(Give sufficient detail, such as suggested above.)

Nature of review contemplated in the formal processes --

Nature of review contemplated in concrete problems and the like --

Special phase of unit of instruction in any proposed new work --

Additional remarks concerning new work ----------------

Lesson plans. Student teachers should prepare one or more lesson plans during the semester. The chief purpose of the lesson plan is to secure for the teacher an organization of the teaching material. Lesson plans are expected only for topics that require more or less detailed presentation. It is not expected that a lesson plan is to be closely followed in teaching the lesson. Prepare the plan at least one week in advance. This gives opportunity for discussion with your supervisor.

Written work and tests. When the pupils show from the nature of drill work, either at the board or seat, that they know a topic, assign written seat work on it. Pupils may be helped on written seat work that is drill work, but not on written work that serves, either formally or informally, as a test. From day to day there should be systematic drilling on new work and written seat work on the old. A fair distribution of the time is two thirds for drills and explanations and one third for written work. Much depends upon the work and the grade. If proper care is used in these daily written exercises not to assign pupils tasks in which they have been insufficiently drilled, the records in such work should be high. Do not mark pupils on exercises they have not had time to do. Mark papers with the number worked each day and the

number correct. Examples are either right or wrong. Return papers to pupils and have corrections made. File the records of this written work every Friday with your supervisor. Give a weekly test, consuming about two thirds of the class period, in any class in which the daily written work may not have secured the desired review and appraisal of the work. Give a written review test once a month, using the whole recitation period. Individual pupils may be transferred to higher or lower classes at any time. Report to your supervisor pupils whom you think are above or below grade.

Conferences with supervisor. Student teachers should consult the bulletin board daily for notices. Report to your supervisor for special conference if your name is posted. Matters of interest to all student teachers of arithmetic are discussed at the regular general conference.

PREPARATORY TO GIVING UP CLASSES

For the sake of helping both the pupils and the teacher who is to take your class, observe the following suggestions:

1. Make your last weekly outline of work very complete, giving with considerable detail the work covered during the previous week.

2. On the reverse side of the last outline write in detail any work not yet taken up to finish the limit of work for the grade.

3. If the needs of the class are not made full enough in the outline, specify such needs on the reverse side.

4. State the specific needs of certain backward pupils.

5. Help the new teacher in any way you can.

CHAPTER XI

IN RELATION TO THE COURSE OF STUDY

THE PRESENT TEACHING OF ARITHMETIC

Changing ideals. The curriculum of the elementary school to-day is a product of a period of educational unrest. The dominating influence of the three R's began to lessen toward the end of the last century, when the demands of present civilization began to require a readjustment of educational standards. The old education held for mental training. The new education is demanding efficiency. Instead of admitting only those subjects that have been valued for mental culture, the curriculum of to-day must provide for all subjects that make toward better social development. Changing educational aims necessitate a restatement of educational values. The material of the subjects that have a traditional place in the curriculum of the elementary school must necessarily be examined as to its present worth. Parts of the subject matter of arithmetic, for example, that were of use in the past, are of little or no value to-day. The work of eliminating waste material is not confined merely to topics but to unusually complex or difficult exercises as well. This elimination of waste has been going on for a decade or more, and, as a result, the most progressive text in arithmetic to-day is quite a different work from the book

of twenty years ago. Efforts are also being made to enrich the course in arithmetic through more practical exercises, especially where they may be related to the community life of the pupil.

Losses *vs.* gains. A natural result attends these periods of reform. In breaking up any established system and seeking a new adjustment, losses must necessarily accompany gains. The teaching under the old education was thorough. The teacher was effective in drill. That which the pupils knew they knew well. The influences and conditions that accompany the present movement, the pressure of a crowded curriculum, and the recommendations of progressive educators, not to mention the exaggerations of pseudo-reformers, have made it possible for teachers to become less effective in drill and in thoroughness. That there has been less effective work in these directions can hardly be denied. Possibly the "reform" textbooks in arithmetic have been largely responsible for this condition. Among other departures, we had, fifteen or twenty years ago, a new plan, the spiral. The extreme spiral plan has given opportunity for a falling off in thoroughness, especially with respect to the upper grades. The efforts of recent texts to seek a better balance between the topical and spiral plans argues both for thoroughness and effectiveness. There is also need, while the subject matter of arithmetic is being critically examined as to its value, of teachers retaining a pride in their own proficiency in subject matter. It is just as important to-day for the arithmetic teacher to be a capable mathematician as it was a generation ago.

A period of transition. The teaching of arithmetic in the United States has been in a transitional stage during the last fifteen or twenty years. This has been due in large measure to the influence of changing ideals in education. A prodigious number of new texts in arithmetic has appeared within the last decade with solutions for the right teaching of the subject. Some of these books are "extreme," but most of them show progress in the right direction. The efforts to improve the teaching of arithmetic are wholesome and will undoubtedly bear fruit, but the effect on the schools at the present time seems to be questionable. Teachers are vitalizing their work by beginning to eliminate the unusable and to relate arithmetic more effectively to life, but we may well ask if they are maintaining a high standard on the computation side of arithmetic. There is need of greater emphasis on accurate calculation. There is also need of a return to the ideals of Warren Colburn with respect to the teaching of mental arithmetic.

The question of subject matter. The demand is being made that the arithmetic of the elementary school should be more practical, in order that the pupil may see a satisfying relation to his present living. One reason why this cannot be brought about is that the course is padded with a superabundance of worn-out or useless material, either in the form of antiquated problems or unnecessarily hard or unusual exercises. Over twenty years ago President Francis A. Walker in his Boston address on "The Arithmetic of the Primary and Grammar Grades" sounded a note of warning against the current practice of teaching topics and examples

that give neither profit nor pleasure to pupils. His ideas have since been concurred in by others interested in the reform of the teaching of arithmetic and, as a result, the texts of recent years are omitting a vast amount of waste material.

The following material should be omitted in a course for the elementary school:

To be omitted:

1. The long method of finding the greatest common divisor. All practical examples in G.C.D. can be solved by factoring.

2. Complex fractions; and other unpractical fractions, such as those having very large denominators.

3. The tables in denominate numbers that are rarely used, such as troy weight, apothecaries' weight, and apothecaries' fluid measure.

4. Unpractical examples in denominate numbers, like: Multiply (or divide) 8 hhd. 3 bbl. 17 gal. 3 qt. 1 pt. 3 gi. by 3. In any case, hogsheads, barrels, and gills should be omitted.

5. True discount, because it is not used in business.

6. Cube root, compound proportion, annual interest, equation of payments, partnership involving time, and alligation. These topics are omitted in nearly all of our recent texts. Omit the first three because they have no practical value for the average citizen; the next two, because they have no place in modern business methods; the last, because it has long since been obsolete.

7. Traditional problems that have no relation to business.

Among these are the "inverse" examples of the type: Given the principal, interest, and rate, to find the time.

The value of some of the topics of arithmetic is questionable when taught in the traditional manner.

To be modified:

1. Teach only those rules for partial payments that relate to the business practices of your own state.
2. Profit and loss should have no claim as a separate topic.
3. Problems in commission and brokerage in which the agent takes out his commission before buying have no practical value.
4. Omit examples in stocks and bonds that involve fractional shares. Omit the traditional textbook examples, for the average citizen is not concerned with the arithmetic of the stock and bond exchanges.
5. Teach exchange to the extent that one may learn how to send money from one city to another. The methods of the commercial houses do not concern most people.
6. Time discount in discounting notes is unduly emphasized.
7. Longitude and time should be correlated with geography.
8. The metric system should not be taught unless in relation to the work of the laboratory. It is not necessary from the practical point of view in everyday life.

In omitting, modifying, or adding to the subject matter of arithmetic, schoolroom instruction should be influenced by the locality. California schools can well omit the table

of dry measure, for the bushel and peck measures are practically unknown to them. It is possible that troy weight should be taught in a gold-mining community. The value of teaching rods may well be questioned, even in rural schools, for present-day surveyors make all their linear measurements with a tape or chain 100 ft. or more in length; and use 5280 ft. (not 320 rods) for a mile and 43,560 sq. ft. (not 10 sq. chains) for an acre.

The fundamentals in arithmetic. While it is necessary thus to vitalize the subject matter of arithmetic, it is equally important to emphasize the fundamentals of the subject. Above all, arithmetic is the science of calculation, and should continue to be recognized as such. Hence we need to teach children to add, subtract, multiply, and divide, and to do so with facility. The other topics indispensable in calculating are factors and multiples, including greatest common divisor and least common multiple; common and decimal fractions; ratio and proportion; and square root. Percentage is indispensable, but it adds nothing new to the operations in common and decimal fractions except its symbolism. The rest of a course in arithmetic consists of measurements and applied problems.

THE AIMS IN TEACHING ARITHMETIC

Statement of aims. Various reasons have been given for the teaching of arithmetic. It is held, among other things:

1. That the study of arithmetic should sharpen the wit.
2. That it should afford mental discipline.
3. That it should prepare for further work in mathematics.

4. That it should give accuracy and facility in computation.

5. That it should identify the child with life.

While the first three of these aims may have some claim to recognition, it has been their dominating influence that has helped to make arithmetic a drudgery to the child and an ineffective tool in his mental equipment.

Fundamental aims. We shall assign the last two aims as the fundamental ones. Let us then give the pupil:

a. Skill and accuracy in arithmetical calculations.

b. School experiences that fit as far as possible for life's experiences.

Of these two aims, the first assigns the primary reason and the second the ultimate reason for the study of arithmetic. The first is of such importance that it should not be forgotten while we are seeking fruitful applications of arithmetic that pupils must learn to "figure." Each of these aims should supplement the other.

Question

Why should the aims discarded above be no longer retained as fundamental?

STANDARDIZING RESULTS IN TEACHING ARITHMETIC

From the standpoint of theories and methods of teaching arithmetic, teachers of to-day are apparently working in the light. One agrees that such and such doctrines are sound and proceeds to organize his work accordingly. But the appraisal of the results of the work of individual

teachers and of school systems is largely based on impressions. It is only within the last few years that any scientific attempts have been made to answer questions like the following: How does any particular class or system of schools rank in arithmetical ability in comparison with the representative school systems of the United States? How does the ranking in the fundamental operations in arithmetic in any school or system of schools compare with its ranking in reasoning power in problems? What relation exists between the time spent on arithmetic and the resulting abilities? To what extent does the course of study influence the resulting abilities? It is outside the plan of the present book to consider these and related questions. For a scientific treatment of such problems, the reader is referred to the recent statistical study, *Arithmetical Abilities and Some Factors determining Them*, Columbia University Contributions to Education, Teachers College Series, by Cliff W. Stone. Some of the conclusions of Dr. Stone with respect to the work of the first six grades in twenty-six representative school systems in the United States may be briefly summarized: The term "ability in arithmetic" should be better expressed as "abilities in arithmetic"; a high ranking of a pupil in the mechanical operations does not guarantee a high ranking in reasoning and *vice versa;* neither is there a close correlation between abilities of pupils in some of the fundamental operations; namely, addition and subtraction. A study of the relations between time expenditures and the resulting abilities and between the course of study and abilities is carefully considered by Dr. Stone. Any teacher

who is not familiar with the attempts being made to "test hypotheses" and to harmonize opinions and facts in the teaching of arithmetic should read the above-mentioned book, which, although it requires close study, can be readily understood even by one who is not familiar with modern statistical methods in the study of educational problems. Perhaps one of the most satisfying results that any teacher or superintendent of schools may obtain by putting into practice some of the suggestions given by Dr. Stone is to find out how the pupils in his own school or system of schools rank in abilities with those of some of the representative systems of the United States.

THE CHARACTER OF PROBLEMS FOR THE ELEMENTARY SCHOOL

Basis for selection. The pupil's interest is the main thing to be considered in selecting problems for the lower grades, the question of ultimate value being of less relative importance, especially in the earliest work. The using of objects as a basis for story problems and the playing of number games serve admirably to stimulate the interest of a class. In the upper grades, the problems should have real applications in life and come within the experience of the average citizen. When the right selection of problems is made, the question of the pupil's interest is probably solved. Some of our newest texts use the plan of classifying problems under headings such as earning money, marketing, purchases, and industries. The plan is excellent, but there is need of the teacher's adapting such work to local conditions and to the needs of individual

classes of the school. For instance, the text may give a set of problems relating to the mining industry, while the school in question is located in a section of the country far removed from any sort of mine. It is clear that the emphasis in such a case should be, not on the mining industry, but on some industry peculiar to the community. Problems on household purchases, earning money, and the like are unquestionably excellent for any school.

Problems for the primary grade. The problem of the first two years is usually the story problem that arises from some kind of object work. The interest stimulated in the children is determined, not only by the character of the problems, but by their wording:

Ex. Clara had 5 little bunny rabbits in a pen and 3 squeezed through the fence. How many bunnies did Clara have left ?

A less interesting statement with the same numbers is:

Ex. 5 boxes less 3 boxes are how many boxes?

After the class have arranged their blocks to look like trains of cars, ask this:

Ex. How many cars are there in your train, Charlie ? If $\frac{1}{3}$ of the whole train should fall off the track, how many cars would remain on the track ?

This one will not be so attractive:

Ex. If I take away $\frac{1}{3}$ of the whole number of blocks, how many blocks are left ?

This mental problem is suitable for the third and fourth grades:

Ex. Suppose, Orval, that you sold 9 papers at 5 cents each. How much money would it be necessary for you to add to the amount received from the sale of papers in order to be able to buy a toy engine valued at $1 ?

This is less stimulating:

Ex. If one hat costs $5, how much will 9 hats cost?

This and the three following problems form a chain of problems relating to earning money:

Ex. Arthur works in his uncle's store 1 hour each day after school and 6 hours on Saturdays. How many hours a week does Arthur work for his uncle?

Ex. If Arthur's uncle pays him 15 cents an hour, how much does Arthur receive in 1 week? In 4 weeks?

Ex. Arthur also earns each week 35 cents for errands and $1 for selling papers, out of which he pays 4 car fares at 5 cents each. What is his net profit each month from these two enterprises?

Ex. What is Arthur's total income for 4 weeks from his several lines of work? For one year of 52 weeks?

Problems for the grammar grade. It is refreshing to note the changing character of the problems for the upper grades as shown in some of our recent texts. The old stock problems on the cost of hats, cows, and horses, not to mention those on filling cisterns and other problems from the time of the Middle Ages, are not so conspicuous as formerly, and we have, instead, those of livelier interest, some of which relate to household purchases and to the most common investments. The solution of the difficulty of making arithmetic problems practical lies in relating individual problems to larger and more vital ones. The idea of a chain of problems, illustrated above, comes near to the solution.

A chain of problems in household expenses may be worked after getting local prices: Find the cost of a 12½-pound turkey; a piece of bacon weighing 8½ pounds; a ham weighing 11 pounds; a porterhouse steak weighing

$1\frac{1}{2}$ pounds. Find the cost of other amounts of provisions and determine how much it will cost a family of five for provisions for one month. Data may be supplied by the teacher and pupils. Other household expenses may be considered and, the income of the family being known, the amount of money saved per month may be determined.

If the school is in a rural community, the class could find a variety of problems in connection, say, with a study of the wood business. Consider the cost of purchasing standing timber and the expense in felling, cutting, hauling, and piling the wood. Find the profit after selling the wood at retail. Valuable data may often be gained, both from the local data and from the solution of such problems.

Again, the class may go through the forms of certain business transactions, learning thereby proper business methods and the necessary arithmetic involved. If actual business can be created in relation to the activities of the school, much better results can be obtained. A school bank may serve as a center about which such lines of work may be grouped.

A very suggestive article on "Community Arithmetic for the Seventh and Eighth Grades," by Walter W. Hart, may be found in *The Elementary School Teacher*, University of Chicago Press, Vol. XI, No. 6.

Questions

1. Gather data for a chain of problems relating to the wood business or some other industry.

2. In what specific ways may the course in arithmetic be enriched through the self activities of the pupils?

THE CURRICULUM IN RELATION TO THE PUPIL

Counting and sense training period. It is almost an instinct with the child to count. Since he enjoys its rhythm, let him begin his study of arithmetic by serial counting, by which we mean the counting of numbers in their natural order, 1, 2, 3, 4, etc., and by 10's, 5's, 2's, etc. The child recognizes the number of objects in a group first by counting. He may later recognize as many as five in a group without counting, or more, if the objects are regularly arranged like the dots on dominoes.

The child of the primary grade is particularly interested in the world of objects. He does not consciously analyze or generalize, but learns largely through perception. In recognition of this fact, much of the early work should be associated with sense training. Let the child measure, cut, build, and draw. Opportunity is thus offered for the comparison of magnitudes, so that one aspect of number, the ratio idea, is developed. Kindergarten work is especially valuable in this connection.

Reflexive or memorizing period. Sense training, usually in the form of measurement, continues throughout the earlier grades. Out of this and out of serial counting and number grouping the pupils gather much information concerning the relations between the different numbers. Now the forty-five combinations must be memorized, columns must be added, subtraction mastered, and the multiplication tables learned by heart. To accomplish all this the pupil must work, for the most part, quite mechanically. He is by nature concerned more with the "how" than with the "why."

Reasoning period. Since most of the primary work is of a mechanical nature, the reasoning power of the pupil is rarely called into play, but in the later grades, where the work demands a more logical treatment of the subject matter, he gradually becomes more reflective than reflexive and is expected to give more attention to serious reasoning.

Questions

1. Of what value is visualizing in connection with sense training?
2. What work should be learned reflexively by the pupil?
3. Name three fairly well-defined stages in the development of the pupil's mathematical powers.

THE CURRICULUM

Present tendencies. There has been a tendency in recent years to shorten the time given to arithmetic. The Committee of Fifteen recommended (1895) that much of the material in the course of arithmetic be eliminated and thus a corresponding shortening of time be secured. The committee stated that five years should suffice, beginning with the second, and that the seventh and eighth years should be given over to algebra, taught as generalized arithmetic. The demands of new subjects for a place in the school program have added another argument for shortening the time formerly devoted to arithmetic and the older school subjects.

The schools have in varying degree generally adopted the recommendations of the Committee of Fifteen regarding the use of algebra in the last two years. The average school to-day, it may be safely stated, restricts itself to the use of x in the solution of problems. Many schools

also teach simple geometric principles and constructions. There is also a quite general agreement that formal drill work in arithmetic should not be begun before the second year. But the total time devoted to arithmetic in the majority of schools is perhaps nearer eight than five years, as recommended by the committee.

Factors in shaping the curriculum. With respect to arithmetic, as with other subjects, the curriculum of the elementary school should be shaped with reference to:

1. The interests and mental development of children.
2. The leading aims that dominate the teaching of the subject.
3. The subject matter.

The curriculum is the product of the proper relating of these three factors.

Lines of emphasis. The lines of emphasis in a course of arithmetic naturally correspond to the chief teaching aims. The curriculum should therefore provide, first, for much calculating, whose sequence follows in the main the logical nature of the subject matter. Secondly, it should provide for applications of the calculating side of arithmetic, both with respect to the work of the school and to the life outside the school. The application side of arithmetic is emphasized in the early work in the various aspects of measurements and in problems of an objective nature. The study of measurements in the early grades is the beginning of the work in denominate numbers, which continues through several years.

Topical *vs.* spiral arrangement of subject matter. In recent years the schools have abandoned the strictly topical

plan, whereby one subject is exhausted before another is taken up, and have adopted to a greater or less extent the spiral plan, whereby a number of topics are covered two or more times, first considering the simpler aspects of each topic, and then with each turn of the spiral bringing in harder phases. The pupils may work, for instance, simple examples in addition followed by simple examples in subtraction before taking up column addition. Simple multiplication and division are usually taught before the hardest case in subtraction.

The spiral plan seems especially well adapted for the early work, where concentration of effort is not possible and where variety is necessary. One advantage of the spiral plan is that pupils who leave school early cover a wide range of topics. An advantage of the topical plan is that it permits concentration of effort. The spiral plan may bring an opposite result. Experience is proving that neither the extreme topical nor extreme spiral plans are satisfactory. There is need of a proper balance between the two. It would seem wise to plan the work of the earlier grades with the emphasis on the spiral arrangement of subject matter and that of the later grades with the emphasis on the topical.

AN OUTLINE OF THE COURSE OF STUDY IN ARITHMETIC

The first two years. The general tendency in this country is to refrain from any drill work in the first year. In fact, some schools defer such work until the middle of the second year or the beginning of the third. The first half of the first year should require only sense training,

counting, and reading and writing numbers demanded by the reading lessons. As the work proceeds during the year, emphasize as fundamentals:

a. Counting by 1's, 10's, 5's, and 2's. Recognizing the position of numbers in the number series. Reading and writing numbers from 1 to 120, or possibly to 1000.

b. Sense training, including number grouping as on dominoes, paper folding, measuring, and other forms of "doing" problems.

Before the second year is finished, the pupils should have in addition to the above an understanding at least of the following:

Reading and writing numbers in the space 1–10,000.

Comparative magnitudes, including the foot, the inch, and the yard; the pint, the quart, and the gallon; the five-cent piece, the dime, and the dollar; special attention to the fractions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$.

Roman numerals up to XII. Telling time.

Familiarity with addition, subtraction, multiplication, and division in the space 1–20 in relation to objects and measuring work.

Memorizing those of the forty-five combinations whose sums are not greater than 10.

Addition and subtraction in the higher number spaces, both oral and written, like

$$\begin{array}{r} 60 \\ +\underline{10} \end{array} \qquad \begin{array}{r} 80 \\ -\underline{10} \end{array} \qquad \begin{array}{r} 55 \\ +\underline{\ \ 4} \end{array} \qquad \begin{array}{r} 49 \\ -\underline{\ \ 5} \end{array}$$

The next four years. Before column addition is taught, the pupils have passed over the first turn of a spiral, having

learned something of addition, subtraction, multiplication, division, and the simpler fractions.

With the second turn of the spiral, during the latter half of the second year or the beginning of the third, they begin in a more serious way the study of the four fundamental operations, starting with column addition. While the order of teaching is addition, subtraction, multiplication, and division, the strictly topical plan should not be adhered to. Teach the inverse operations in close connection with the direct operations: subtraction with addition, and multiplication with division. When the pupil learns his multiplication table of 4's, let him apply it by multiplying by 4 and then dividing by 4, etc.

The study of common fractions, begun in a simple form in the first two years, parallels the work just described. Work in denominate numbers also finds a place in the early grades, beginning even in the first grade. Decimal fractions should be introduced before common fractions are thoroughly mastered.

By the end of the sixth grade the pupils have learned the computation side of arithmetic, with the exception of square root and ratio and proportion. The sixth grade should provide for extended reviews covering the work of the previous grades, especially in the mechanics of the fundamental processes as applied to whole numbers and fractions, both common and decimal.

Beginning with the third grade and ending with the sixth grade, the computation side of arithmetic includes:

Notation and numeration (emphasizing place value).
Roman numerals.

Memorizing the 45 combinations.

Column addition.

Subtraction (involving "carrying" or "borrowing").

Multiplication tables and written multiplication.

Short and long division.

Factoring, G. C. D., and L. C. M.

Common fractions.

Decimal fractions.

Cancellation.

Percentage symbolism. Relation to common and decimal fractions.

The application side includes:

Denominate numbers, including developing and applying the table of United States money and the tables of length, weight, capacity, and time.

The areas of the simpler geometric figures.

Board measure, carpeting, plastering, and the like.

Problems that relate to the affairs of *everyday life*. The arithmetic of the school garden. Household purchases. Keeping of a cash account. The related business forms. Common industries. Simple aspects of commercial discount and interest.

The seventh and eighth years. By the seventh grade, the pupil should be prepared to understand more fully the reasoning involved in arithmetic, and since the formal side of the subject has been practically covered, there is opportunity for this development. The new processes on the computation side for the seventh and eighth years are:

Ratio and proportion.

Powers and square root.

The application side includes:

Mensuration of familiar plane figures and solids. Opportunity is found here for the application of square root and ratio and proportion.

Longitude and time, taught in its relation to geography.

Applications of percentage — without time: profit and loss, commercial discount, and commission and brokerage.

Applications of percentage — with time: simple interest, discounting notes, partial payments, and compound interest.

Other applications of percentage: insurance, taxes and duties, stocks and bonds, and exchange.

Partnership without time. Opportunity is found here for the application of ratio.

Conclude the work of the eighth grade with a review of the most essential parts of arithmetic, including especially the fundamental processes applied to whole numbers and common and decimal fractions. Include also in the finishing work of the eighth grade a review of the business side of arithmetic, such as writing notes, bills, receipts, checks, and other forms of business paper, and the keeping of cash accounts and accounts with persons as a minimum requirement in bookkeeping for the elementary school.

Algebra. The *algebraic method* is of great service in solving many of the problems of arithmetic, especially in percentage, and hence free use should be made of the equation. The "x" should also be used in proportion. Aside from these applications and, possibly, exercises in finding the values of algebraic expressions by substituting

numerical values for the letters involved, the subject of algebra should be reserved for the regular high school course.

Geometry. Practical geometry is utilized in mensuration. Simple constructions by means of compasses, triangles, and protractor may well be related to this work. It is well also, if time permits, to develop by the inductive method the properties of the most common geometric figures. In case any special emphasis is laid on concrete geometry, it is better to give it a place on the program separate from arithmetic.

Importance of mental arithmetic. The curriculum should provide for a great deal of mental arithmetic, which should be emphasized at every opportunity in connection with written arithmetic. The first five minutes of the recitation period should, as a rule, be devoted to sharp mental drills of a review nature.

Importance of addition. The importance of addition should not be underestimated. No one should lay claim to skill in arithmetic who is not quick and accurate in addition. One who cannot add well cannot be good in computation. Drills in mental and written addition should be continued throughout the grades until the pupils are accurate and fairly rapid.

Questions

1. Why would it be difficult for schools to adopt the suggestion of the Committee of Fifteen regarding the number of years that should be given to the study of arithmetic?

2. What ideas should control the shaping of a course in arithmetic?

3. Discuss the arguments for and against the spiral and topical plans with respect to the earlier and later grades.

4. Consult the course of study of some representative city and ascertain the ground covered in arithmetic for each of the eight grades. Note the lines of emphasis.

5. What is the place of algebra and geometry in the grades?

6. What is the advantage in teaching mental and written arithmetic at the same period?

DETAILED COURSE FOR THE FIRST YEAR AND A HALF

The following detailed course in early number work, which is being used in the elementary department of the author's own normal school, presents a well-defined sequence of teaching material for the first year and a half, or until the pupil is ready to learn systematically the fundamental operations in arithmetic.

Low First

The chief aim is the preparation for column addition, which is to be taught a year and a half later, not by teaching now the addition combinations either with or without objects, but by teaching the children to know the places of the numbers in the number series. This presupposes counting and a knowledge of the number symbols.

The teacher will find the following outline of work for the Low First arranged so as to give an easy and logical development of the facts to be learned with respect to the place of numbers in the series. This arrangement appears below under the captions: counting, number groups, reading and writing numbers, and the place of numbers in the series. It is along this line that pupils are to be systematically drilled throughout the term.

The other work that enters into the Low First is that usually given under the headings of Sense Training, Busy

Work, Measuring, and Comparisons. The order in which this is introduced may be varied by the teacher as occasion demands. It is in such work as this that pupils learn to express their thoughts in simple language based upon experiences gained within or without the class. Only the simplest phases of the matter treated should be introduced in the Low First. Teachers should not be misled into magnifying the extent of the work expected in this line on account of the variety of topics presented below. Of the two lines of work mentioned in this and the preceding paragraph, that outlined in the first should receive chief attention. Much time is especially needed in writing numbers.

1. *Relative position, direction, magnitude.* Call on individual pupils and, in certain exercises, on the whole class. Which is your *right* hand? Which is your *left* hand? Your *right* eye? Place your *left* hand on your *right* arm. Who is on the *right* of Mary? On the *left* of John? Point to your *right*. Your *left*. *Behind* you. *In front of* you. Write the figure 2 on the board. Write another 2 *above* the first 2. Write another *below* the first 2. Write a 1 on the board. Write another 1 close to this and *to the right of* it. Write another 1 close to the first 1 and *to the left of* it. Point to the *top* of the blackboard. Point to the *bottom* of the blackboard. Mary will stand to the *left* of John. Julia will stand *between* John and Mary. Place your finger on the *middle* of this ruler.

   Which of these blocks is the *largest?* Which is the *smallest?* Which is *longer*, this ruler or that ruler? Who is the *tallest* pupil in the class? Who is the *shortest?* Who is *taller*, Mary or John? Who is *shorter*, Ida or Sue? The class will stand in a row. Who is the *tallest?* He may stand first in the row. Who is next to the *tallest* in the class? He may next take position. The pupils take position according to their respective heights. Compare weights of objects at hand.

2. *Counting:* 1–12; 1–20. With and without objects. There is little need of counting with objects beyond 20. Have pupils count things they are interested in. How many pupils are there in the class this morning? How many boys? How many girls? How many feet has a horse? Name other animals that have four feet. John may make three marks on the board. Mary may make some marks on the board. Sue may tell how many marks Mary has made. See that the place of number in the series is not confused with number itself. Ask Mary, for instance, to fetch the first three of a number of blocks placed in a row. Then ask her to replace them and bring the third block in the row. Who is the third pupil from the left in line? Who is the second from the other end? The abacus furnishes a convenient means for counting with objects.
3. *Number groups.* Unless the objects are specially arranged, pupils cannot recognize more than four or five objects in a group. Teach the recognition of number groups like those on dominoes. Have flash cards with spots thus arranged. Place the pupils two in a group and have them count the number of groups. The abacus is handy for this kind of work, where the aim is to prevent the idea of a fixed unit. How many 3's do you see? How many 2's? How many 1's? (This refers to objects in groups, not to the number symbols.) Do not yet count serially by 2's, 3's, etc.
4. *Reading and writing numbers:* 1–12; 1–20. Board work only. No pencil throughout the term. Relate the number (found by counting), the name, and the symbol (the figure). Read numbers written by the teacher. Use flash cards to teach quick recognition of the figures.

   Pupils write numbers at board under models set by the teacher. The crayon should be held not tightly, yet firmly. Short pieces of crayon are desirable for small hands. Aim toward large rather than small figures. Where pupils have difficulty, let them trace over the copy made by the teacher. Then let them try without tracing. The teacher makes a 3 on the board. She asks the class to look at it carefully. She erases the 3 and asks the class to make one like it. Write 13 on the board. Which figure is on the right side? Which is on the left side? Which figure do we make first in writing 13? In reading the "teens" the teacher should place the pointer first on the right figure as the word is

pronounced. This will help to prevent the writing of 13 with the 3 first. Much individual attention should be given the pupils while they are learning to make the different figures. Be sure that the 8 is made toward the left, beginning at the top. Write the numbers in columns.

5. *Counting:* 1–50.

6. *The place of the numbers in the series:* 1–10; 1–20. Extend the limits of the series as indicated. The teacher should have on the board the series to be studied. The numbers 1–50 and 51–100 may be printed on strong linen or heavy paper. A strip 9 feet long and 6 inches wide will answer the requirements for figures $1\frac{5}{8}$ inches high. A set of stamps for figures of this size costs about 50 cents. The chart is to be used in the following exercises. Do not force the children to visualize the figures. With the chart on the wall for their daily study, the power of visualizing will follow easily and naturally.

(*a*) Name the number next after 3, 7, etc.; and, later, after 13, 19, etc.

(*b*) Name the number before 6, 10, etc.; and, later, before 16, 20, etc.

(*c*) Name the number between 5 and 7, 12 and 14, etc.

(*d*) Name the numbers between 4 and 7, 15 and 18, etc.

7. *Reading and writing numbers:* 1–50.

8. *Time measures.* What day is to-day? What day was yesterday? What day will to-morrow be? Name the days of the week. How many days are there in one week? How many school days? How many working days? What month is this? In what month is your birthday, Sara? What day of the month (date) is your birthday? What day of the month is to-day?

At what hour in the morning does school begin? At what hour is school dismissed in the morning? When does school open in the afternoon? Refer to the clock face. The teacher draws a figure of a clock, using Arabic numerals. Where is the hour hand at 9 o'clock? Where is the minute hand? Sara was 5 minutes late this morning. Where was the minute hand? Do not go beyond the "time" that is of interest to the pupils.

Compare the ages of the pupils.

9. *Sense-training.*[1] Review relative size, as previously taken up,

[1] See p. 34, remembering that not all applies here.

thereby training the sight judgment of the pupils. Train judgment of relative size through the sense of touch, the eyes being closed. For both of these purposes use various kinds and shapes of objects. Compare weights of objects. Thus far the pupils are not supposed to make exact comparisons, although the "one half" and "two times" may be brought out.

Train the pupils in the power of visualizing objects, the aim being to have them recall the properties of the objects and the number of facts involved. Write a number on the board. Erase it and ask for the number erased. Write two numbers on the board. Erase them and ask for the numbers erased. Ask questions about the position of the figures in certain numbers and see if the pupils have the correct mental picture. Test with 13 and 31.

10. *Counting and reading and writing numbers:* 1–50. Review previous work. How many figures are there in 25? In 6? Have pupils point out numbers on the chart as they count. Point out on the chart numbers that have been named. In looking for 43, for what number do you first look? Point out numbers on the chart and have them named. How many numbers are written on the chart? Children count and discover that the last number written in the series tells how many have been written.

Ask the pupils to write the first 5, 10, or 15 numbers in a column without reference to the chart. Ask them to begin with a certain number and write two, three, five, or ten more. Count by 10's and by 5's.

11. *Comparative magnitudes.* Review previous work. Compare both unequal and equal magnitudes, using a great variety of objects and drawings. Most of the drawing should be done by the teacher on the board, — the square, the rectangle, and the circle being employed. The pupils may be given a number of blocks of equal sizes, from which they may build larger solids that will furnish means of comparison. Colored squares made out of cardboard will interest the children and serve a similar purpose. These objects and drawings will give sufficient material for bringing out the relations "one half" and "two times" and perhaps a few other simple fraction facts.

Develop the need of measuring to determine equality. Relate here the foot and the yard (Numbers 1–3) and the inch and the foot (Numbers 1–12). Refer to Speer's "Primary Arithmetic

for the Use of Teachers" but keep in mind the fact that the work in comparing magnitudes there given extends over more ground than is contemplated for the Low First. Most of the material on pp. 37–60 of that text can be used here.

12. *The place of the numbers in the series:* 1–50. Use the chart with numbers written from the bottom up. Encourage the pupils in visualizing the numbers on the chart. After the position of the numbers is fixed in his mind, the pupil naturally can answer the teacher's questions more readily when not looking at the chart than if required to examine it.

(*a*) Drill as in (*a*), (*b*), (*c*) under (6) above. Emphasize especially, asking for the numbers next after 9, 19, 29, 39, 49, and the numbers immediately before 50, 40, 30, 20, 10.

(*b*) The decades. After the first ten numbers come the "teens." After the "teens" come the twenties. After the twenties come the thirties, etc. Before the twenties come the "teens." Before the thirties come the twenties, etc.

(*c*) Step halfway across the room and then stop, Charlie. Locate the middle of this stick, Mary. Point out the halfway point of this vertical line on the board, Arthur. Find the number on the chart that is halfway between 10 and 20; halfway between 20 and 30, etc. Is 23 nearer 20 or 30? Is 37 nearer 30 or 40, etc.

The number table may be extended to 100 for counting, reading, and writing numbers, and locating numbers in the series in case the pupils are ready. The counting and the reading and writing of numbers to 100 can be done almost as soon as they can be done to 50, but the locating of the numbers in the series 50–100 had better be left for the High First.

High First

The work of the High First is a review and continuation of the work of the previous term. The main emphasis should be on the reading and the writing of numbers and on the locating of numbers in the series. The study of comparative magnitudes is continued. The essentially

new work consists of finding out the number facts in connection with objects. In this connection the numbers 1–12 are studied, the pupils employing addition, subtraction, multiplication, and division, the latter in its dual form, — division by measuring and division by partitioning. No effort is to be made here to teach the abstract number facts, for, according to the plan of the course, no use is to be made of these facts for a year to come; but it is essential that pupils learn the meaning of these processes in connection with *things*. We may remind ourselves here that an objective understanding of the fact that 9 and 4 are 13 bears no immediate relation to the memorizing of that fact. When the pupil is ready to use the fact that 9 and 4 are 13, he should not be bothered with the so-called reasoning involved. He needs to use it then as a tool. Number stories should accompany this early study of the operations.

The abstract drills, indicated below, in connection with recognizing the place of the numbers in the series should be given in the order as arranged. The rest of the work may readily be rearranged according to the wish of the teacher. Before teaching any topic, review the corresponding work in the Low First.

1. *Counting:* 1–100; 1–120. Count by 1's, 10's, 5's, and later by 2's. Introduce even and odd numbers. Count by naming the even numbers from 2 to 20; from 20 to 50; by naming the odd numbers from 1 to 21; from 21 to 51. Count objects, naming the place in the series. Thus, 1st, 2d, 3d, etc.
2. *Reading and writing numbers:* 1–100; 1–120. Much time should be spent on this work. It is preferable to write numbers entirely on the board. The class may write 37. Yours is good, John. You may now make a straight column of three 37's and be careful not to write too near the vertical line. Correct individual

errors of the pupils. Write the even numbers up to 20. The odd numbers up to 21. Yours is not good, Julia, because your second column is too near the first. Write the numbers 1–20, placing the odd numbers on the left of a vertical line and the even numbers on the right.

3. *Time measures.* Extend the work of the Low First. Tell time to the nearest 5 minutes, first reviewing counting by 10's and 5's as far as 60. Teach the reading (not writing) of the Roman numerals I–XII. Count 5 seconds, 10 seconds. Remain silent 10 seconds. Point out on the clock 5 minutes after the hour, 10 minutes, etc. How many minutes are there in one hour? In a half hour? In a quarter of an hour? How long does it take you to walk home? How much time do we take for recess? For noon?
4. *The place of the numbers in the series.* 1–100; and, later, 1–120. Use the charts referred to in the work of the Low First and follow the plan of (6) and (12).
5. *Comparative magnitudes.* Bring out objectively the ideas of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$. Draw diagrams of squares and rectangles and compare lengths of sides. Compare areas (sizes). Also use blocks. Speer has good suggestions. Give the children squares of paper.[1] Give directions for folding so as to get two equal parts; to get four equal squares. Bring out the $\frac{1}{2}$ and $\frac{1}{4}$ relations. Also twice as large and four times as large.
6. *Counting:* 1–120. Emphasize the decades. Count by 10's beginning with 10, 1, 2, 3, etc. For this purpose have a number table already written on the board, the numbers 1–10 in the first column, 11–20 in the second, 21–30 in the third, and so on up to 100. Read the horizontal rows. Then count without looking at the table.
7. *The place of the numbers in the series:* 1–120. Use the charts as an aid toward visualization.
   (*a*) First explain, for example, what is meant when we say that 24 ends in a 4 or that 16 ends in a 6. What does 34 end in? 47? 50? Find a number among those which I write that ends in a 2. In a 9.
   (*b*) Name the number in the twenties that ends in a 3. In the thirties that ends in an 8. In the "teens" that ends in a 4, etc.

[1] See p. 36.

(*c*) What is the first number after 20 that ends in a 0, etc.? What is the first number after 5 that ends in a 5, etc.? Proceed similarly with the endings of other numbers ten numbers apart.

(*d*) Bring out the ideas of greater than and less than (after and before in the series). Which comes first, 15 or 17, 26 or 29, 38 or 40, 31 or 29? Notice the gradation. The last set of numbers carries the pupil into different decades. Which is greater, 12 or 10, 17 or 14, 23 or 27, etc.?

(*e*) Numbers after 10, 20, etc. that end with certain figures but not ten numbers apart. What is the next number after 10 that ends in a 3, 5, etc.? After 20, 30, etc.?

(*f*) Numbers after any numbers in the same decade. What is the next number after 13 that ends in a 5, 7, etc.? After 23, etc.?

(*g*) Numbers after other numbers, the latter ending in a 0. What is the next number after 16 that ends in a 0? After 26, etc.?

(*h*) From one decade into the next. What is the next number above 7 that ends in a 1? Above 17 that ends in a 1? Above 27, 37, etc.? Follow the same plan with other endings. This drill is very important since the ideas involved are essential in later work in column addition.

8. *Complementary and measure contents of numbers:*[1] Numbers 1–6; 1–12. This work, in which the pupils are to learn the early number facts, is to be *entirely objective*. Use all sorts of objects. Splints, blocks, and the balls of the abacus are good. It is, perhaps, better to teach the complementary contents for any number, that is, the addition and subtraction facts, first; but in concrete work with objects where no effort is made to have the pupils memorize results teachers will find, for example, that a pupil can find the number of 2's in 6 as readily as he can show the teacher with his sticks that 2 and 4 are 6. The addition and subtraction facts are to be taught for all numbers in the limits assigned above. The measure-contents facts, that is, multiplication, division, and partition, are to be taught in close connection with the complementary-contents facts, but do not at this time teach in this connection the prime numbers, 3, 5, 7, and 11.

As a plan of procedure, the following is suggested: Teach the

[1] See pp. 36–39.

complementary contents of the number 6. Here the pupils find from the use of objects that 5 and 1, 1 and 5, 4 and 2, 2 and 4, and 3 and 3 make 6; and that subtracting in turn 1, 2, 3, 4, 5 from 6 leaves 5, 4, 3, 2, and 1, respectively. Then teach the measure contents of 6: two 3's are 6, three 2's are 6; in 6 there are two 3's and three 2's; $\frac{1}{2}$ of 6 is 3 and $\frac{1}{3}$ of 6 is 2. There is no gain in teaching now that six 1's are 6, one 6 is 6, and $\frac{1}{6}$ of 6 is 1. Next teach the measure contents of 4; next the complementary contents of the number 7; then all the number facts about 8 in the way number 6 was studied. Take up the other numbers, as the term advances, in the way that seems best to the teacher.

Teach number stories in connection with the above work. This gives the pupils a chance to cultivate their powers of expression. The number story should follow the objective presentation and be based upon it.

9. *Comparative magnitudes.* Review measurements, using feet, inches, and yards. Let the children make drawings, dividing up squares and rectangles. Teach pints, quarts, and gallons, if not already used in (8). Let the pupils compare the capacities of these measures by filling them with water.

10. *Counting, reading, and writing numbers:* 1–200; 1–1000. Omit the teaching of place value with the exception of naming and having the pupils name hundreds' place. Count and write by 10's and 100's.

Low Second

Review the work of the previous grades. By the end of this grade the pupils should have acquired considerable command of oral expression in the telling of number stories and the like. The placing of the numbers in the series should be continued and emphasized even more than in the previous grade, since in the next grade column addition will be begun.

1. *Counting, reading, and writing numbers:* 1–10,000. Continue serial counting, as in the previous grade. Count by 10's, 5's, 2's (both the series of the odd and the even numbers), 100's and 1000's.

Begin at any multiple of 10 in the series to count in this manner. Begin with 360 and count by 10's to 400. Count by 100's from 300 to 1000. Count by 1000's from 1000 to 10,000. Write the higher numbers in the natural serial order and as above. Spend much time on the reading and writing of the higher numbers. Have both board and seat work. Teach the names of the different orders or places. By the end of the term the pupils should have learned how to build numbers of two places and of three places by using splints and rubber bands.[1]

2. *Comparative magnitudes.* Extend the drawing work for fractional and multiple relations begun in the previous grade. Let the pupils draw diagrams freehand at the board and with short rulers at the seat, showing relations specified by the teacher. For instance, draw a square. Divide it so as to show $\frac{1}{4}$ of it. Again, draw a square and then a second square near the first that shall be 4 times the size of the first square. The fraction symbols may now be introduced but not used in operations. Teach the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$. The pupils learn the relations that exist between $\frac{1}{4}$, $\frac{2}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{4}{4}$. Also between $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{3}{3}$; and between $\frac{2}{2}$, $\frac{3}{3}$, and $\frac{4}{4}$. Relate to partition (see (4) below) with counted objects. Thus, find $\frac{1}{3}$ of 12 counters; $\frac{2}{3}$ of 6 blocks.

3. *Place of the numbers in the series.* Review the work of the previous grade, extending the numbers beyond 120. The number charts (1–50; 50–100) should be on the wall for reference. Among other things to be emphasized is the naming of the number following another number and having a stated ending. For instance, ask for the next number after 83 that ends with a 1. This is vital in column addition, for, when the pupil has learned that 3 and 8 are 11 and that 8 added to any number ending in a 3 gives a number ending in a 1, he is in a position to tell that 91 is the sum of 83 and 8.

4. *Complementary and measure contents.* Review and extend the work of the previous grade. Carry the study of the numbers, still entirely objectively, to 20. Make no attempt to have facts memorized. Work of this kind is not intended so much to prepare for the later learning of the number facts, because this would be idle, but it is given so as to afford the pupils a first-hand application

[1] See pp. 40–42. Do not make the work too exhaustive here.

of the four fundamental operations of arithmetic to things. See the suggestion at the end of (2) above. In teaching the measure contents, consider, first, exact multiples and divisors. For instance, in 12 there are three 4's. Later, in 14 there are three 4's and 2 "over." Continue the use of number stories. While it is suggested that the work under this topic be carried as far as 20, it is not intended that the work should be made scientifically exhaustive. The teacher should keep in mind, above all things, the interest of the children. The abacus will be found very helpful in such work.

5 *Addition and subtraction that is permissible by means of counting.* It is generally agreed that children should not add by counting. In the next grade they will learn that 4 and 5 are 9 merely as a fact, not being permitted to count 5 more beyond 4 to get the result. The addition and subtraction of 1, 10, and 5 as shown below is permissible and desirable:

(*a*) Add and subtract 1. What are 9 and 1, 19 and 1, 29 and 1, etc.? What is 1 less than 10, 20, 30, etc.?

What are 4 and 1, 14 and 1, 24 and 1, etc.? What is 1 less than 5, 15, 25, etc.?

Then give miscellaneous exercises without reference to the decades: 7 and 1, 23 and 1, 87 and 1; 1 less than 7, 16, 93.

(*b*) Adding to numbers ending with a 0, with the corresponding subtractions. What are 10 and 3, 20 and 3, 30 and 3, etc.? What are 10 and 9, 20 and 9, etc.? What are 10 and 10, 20 and 10, 30 and 10, etc.? What is 13 less 3, 23 less 3, 33 less 3, etc.

(*c*) Adding 10 to numbers, with the corresponding subtractions. What are 3 and 10, 13 and 10, 23 and 10, etc.? What is 10 less than 13, 23, 33, etc.?

(*d*) Adding 5 to numbers ending with a 5, with the corresponding subtractions. What are 5 and 5, 15 and 5, 25 and 5, etc.? What is 5 less than 10, 20, 30, etc.?

All the above questions are based on the pupil's previous experience in serial counting.

Also associate 1 and 9 with 9 and 1, 1 and 4 with 4 and 1, etc. Emphasize the making-up idea in subtraction, which is used later when subtraction is systematically taught. Ask, 9 and what are 10? 4 and what are 5? 20 and what are 25, etc.?

6. *United States money.* See that the pupils can recognize the following coins: 5 cents, 10 cents, 25 cents, 50 cents, and $1. What other name has 25 cents? 50 cents? If a whole cake is worth $1, how much is ½ of it worth? How much is ¼ of it worth? If a whole cake is worth 50 cents, how much is ½ of it worth? Identify 75 cents as ¾ of a dollar. What may be purchased with any of the above amounts? Have the children assemble the coins to show 15 cents, 20 cents, 30 cents, etc.

   Make change as the clerk in the store does. This, of course, must involve only those number facts that the children have memorized; namely, the adding and subtracting of 10 and 5. For this purpose, toy money may be used. Exercises like the following are within the powers of the pupils: "I buy some candy worth 5 cents. I give the clerk 10 cents. He gives me 5 cents in change." "I buy a pound of coffee worth 50 cents. I give the clerk $1. He gives me 50 cents in change." "I buy 10 cents' worth of apples. I give the clerk 25 cents. He gives me 15 cents in change." The teacher may play the clerk and, as in the last example, will give back the change as follows: The teacher says 10 cents; then lays out 10 cents, saying 20 cents; then lays out 5 cents more, saying 25 cents. The pupils should also be taught to do this.

To sum up, the features of the work of the first year and a half that should prepare for the application of the addition facts in the higher number decades, and hence secure for the pupils facility in column addition are.

(*a*) Counting by 1's, 10's, and 5's. Counting by 10's, beginning with any number. Counting the series of even and of odd numbers.

(*b*) Naming the number after and before a number. Naming the numbers between two numbers.

(*c*) Learning the order of the decades. The "teens" come before the twenties, the 50's follow the forties, etc.

(*d*) Naming the number immediately following 9, 19, 29, etc., and preceding 10, 20, 30, etc.

(*e*) Naming the number in any special decade that ends with a 1, 2, 3, etc.

(*f*) Naming which of two numbers comes first in the series and which of two numbers is the greater or less.

(*g*) Naming the first number ending with a 1, 2, 3, etc., that follows some assigned number, especially where the answer introduces the next decade. For instance, name the next number after 38 that ends with a 4.

(*h*) Adding 1 to any number and subtracting 1 from any number. Adding 10 to any number and subtracting 10 from any number. Adding any number less than 10 to 10, 20, etc. Subtracting 3 from 13, 23, etc.; 7 from 17, 27, etc. Adding 5 to any number ending with a 0 or a 5 and subtracting 5 from any such number.

The pupils have perhaps already learned a few of the addition and subtraction combinations in connection with object work, but have not been drilled in them as such.

The new work of the High Second consists of the memorizing of some of the forty-five combinations and applications in addition and subtraction.

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The Curriculum in Mathematics. Vol. VII, Nos. 1–4, and Vol. VIII, Nos. 1, 3, and 4. Also bound in one volume.

The Teaching of Arithmetic. By Professor DAVID EUGENE SMITH. Vol. X, No. 1. Also revised and printed in book form. See above.

The Teaching of Primary Arithmetic. By Professor HENRY SUZZALLO. Vol. XII, No. 2.

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Arithmetic considered as a Utilitarian Study. What should be the Course of Study? By C. W. STONE. Vol. III, No. 8.

Modernized Arithmetic. By Professor GEORGE W. MEYERS. Vol. IV, No. 2.

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For more extended bibliographies, consult the books of Young and Smith, mentioned above.

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